

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.

IMM-NYU 292
DECEMBER 1961



NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

Groups of Homotopy Spheres, I.

MICHEL A. KERVAIRE

AND

JOHN W. MILNOR

(Princeton University)

PREPARED UNDER
CONTRACT NO. NONR-285(46)
WITH THE
OFFICE OF NAVAL RESEARCH
U. S. NAVY

IMM-292
c. 1

New York University
Courant Institute of Mathematical Sciences

GROUPS OF HOMOTOPY SPHERES, I.

Michel A. Kervaire

and

John W. Milnor
(Princeton University)

This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Naval Research, U.S. Navy, Contract No. Nonr-285(46), and Princeton University.

Reproduction in whole or in part permitted for any purpose of the United States Government.

December 1961

1. Introduction

All manifolds, with or without boundary, are to be compact, oriented, and differentiable of class C^∞ . The boundary of M will be denoted by ∂M . The manifold M with orientation reversed is denoted by $-M$.

Definition. The manifold M is a homotopy n -sphere if M is closed (that is, $\partial M = \emptyset$) and has the homotopy type of the sphere S^n .

Definition. Two closed n -manifolds M_1 and M_2 are h -cobordant¹ if the disjoint sum $M_1 + (-M_2)$ is the boundary of some manifold W , where both M_1 and $(-M_2)$ are deformation retracts of W . It is clear that this is an equivalence relation.

The connected sum of two connected n -manifolds is obtained by removing a small n -cell from each, and then pasting together the resulting boundaries. Details will be given in §2.

THEOREM 1.1. The h -cobordism classes of homotopy n -spheres form an abelian group under the connected sum operation.

This group will be denoted by θ_n , and called the n -th homotopy sphere cobordism group. It is the object of this paper (which is divided into 2 parts) to investigate the structure of θ_n .

It is clear that $\theta_1 = \theta_2 = 0$. On the other hand these groups are not all zero. For example, it follows easily from Milnor [14] that $\theta_7 \neq 0$.

The main result of the present part I will be:

THEOREM 1.2. For $n \neq 3$ the group θ_n is finite.

¹ The term "J-equivalent" has previously been used for this relation. Compare [15], [16], [17].

Introduction

The purpose of this study is to investigate the effects of a new educational program on the learning outcomes of students in a high school. The program, which was implemented in the 2023-2024 academic year, aims to improve students' understanding of mathematics and science through a combination of traditional classroom instruction and interactive learning activities.

The study is organized as follows:

Section 2 describes the research methodology, including the selection of participants, the design of the study, and the data collection methods. Section 3 presents the results of the study, and Section 4 discusses the implications of the findings for future research and practice.

The study was conducted in a high school in the city of Istanbul, Turkey. The participants were 100 students in the 10th grade, who were randomly assigned to two groups: the experimental group and the control group. The experimental group received the new educational program, while the control group received the traditional classroom instruction. The data were collected through a series of tests and surveys conducted at the beginning, middle, and end of the academic year.

The results of the study show that the new educational program had a significant positive effect on the learning outcomes of the students in the experimental group. The students in the experimental group performed significantly better than the students in the control group on the tests and surveys. The findings suggest that the new educational program is effective in improving students' understanding of mathematics and science.

The study also found that the new educational program had a positive effect on the students' attitudes towards learning mathematics and science. The students in the experimental group reported higher levels of motivation and interest in learning than the students in the control group. These findings suggest that the new educational program is not only effective in improving learning outcomes but also in enhancing students' attitudes towards learning.

The study has several limitations. First, the sample size was relatively small, which may limit the generalizability of the findings. Second, the study was conducted in a single high school, which may not be representative of all high schools in Turkey.

Despite these limitations, the study provides valuable insights into the effectiveness of the new educational program. The findings suggest that the program is a promising approach for improving students' learning outcomes and attitudes towards learning mathematics and science.

Conclusion

The study concludes that the new educational program is effective in improving students' learning outcomes and attitudes towards learning mathematics and science. The findings suggest that the program is a promising approach for improving students' learning outcomes and attitudes towards learning mathematics and science.

The study also suggests that the new educational program should be implemented in other high schools in Turkey. Further research is needed to investigate the long-term effects of the program and to explore the factors that influence its effectiveness.

(Our methods break down for the case $n = 3$. However, if one assumes the Poincaré hypothesis then it can be shown that $\theta_3 = 0$.)

More detailed information about these groups will be given in Part II. For example, for $n = 1, 2, 3, \dots, 18$ it will be shown that the order of the group θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

Partial summaries of results are given also at the end of §4 and of §7.

Remark. S. Smale [25] and J. Stallings [27], C. Zeeman [33] have proved that every homotopy n -sphere, $n \neq 3, 4$, is actually homeomorphic to the standard sphere S^n . Furthermore, Smale has proved [26] that two homotopy n -spheres, $n \neq 3, 4$, are h -cobordant if and only if they are diffeomorphic. Thus for $n \neq 3, 4$ (and possibly for all n) the group θ_n can be described as the set of all diffeomorphism classes of differentiable structures on the topological n -sphere. These facts will not be used in the present paper.

2. Construction of the group θ_n

First we give a precise definition of the connected sum $M_1 \# M_2$ of two connected n -manifolds M_1 and M_2 . (Compare Seifert [22] and Milnor [15], [16].) The notation D^n will be used for the unit disk in euclidean n -space. Choose imbeddings

1. 1000 13 4 11 12 13 14 15 16 17 18 19

2. 1000 13 4 11 12 13 14 15 16 17 18 19

3. 1000 13 4 11 12 13 14 15 16 17 18 19

4. 1000 13 4 11 12 13 14 15 16 17 18 19

5. 1000 13 4 11 12 13 14 15 16 17 18 19

6. 1000 13 4 11 12 13 14 15 16 17 18 19

7. 1000 13 4 11 12 13 14 15 16 17 18 19

8. 1000 13 4 11 12 13 14 15 16 17 18 19

9. 1000 13 4 11 12 13 14 15 16 17 18 19

10. 1000 13 4 11 12 13 14 15 16 17 18 19

11. 1000 13 4 11 12 13 14 15 16 17 18 19

12. 1000 13 4 11 12 13 14 15 16 17 18 19

13. 1000 13 4 11 12 13 14 15 16 17 18 19

14. 1000 13 4 11 12 13 14 15 16 17 18 19

15. 1000 13 4 11 12 13 14 15 16 17 18 19

16. 1000 13 4 11 12 13 14 15 16 17 18 19

17. 1000 13 4 11 12 13 14 15 16 17 18 19

18. 1000 13 4 11 12 13 14 15 16 17 18 19

19. 1000 13 4 11 12 13 14 15 16 17 18 19

20. 1000 13 4 11 12 13 14 15 16 17 18 19

21. 1000 13 4 11 12 13 14 15 16 17 18 19

22. 1000 13 4 11 12 13 14 15 16 17 18 19

23. 1000 13 4 11 12 13 14 15 16 17 18 19

$$i_1 : D^n \rightarrow M_1, \quad i_2 : D^n \rightarrow M_2$$

so that i_1 preserves orientation and i_2 reverses orientation. Now obtain $M_1 \# M_2$ from the disjoint sum

$$(M_1 - i_1(0)) + (M_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each unit vector $u \in S^{n-1}$ and each $0 < t < 1$. Choose the orientation for $M_1 \# M_2$ which is compatible with that of M_1 and M_2 . (This makes sense since the correspondence $i_1(tu) \rightarrow i_2((1-t)u)$ preserves orientation.)

It is clear that the sum of two homotopy n -spheres is a homotopy n -sphere.

LEMMA 2.1. The connected sum operation is well defined, associative, and commutative up to orientation preserving diffeomorphism. The sphere S^n serves as identity element.

Proof. The first assertions follow easily from the lemma of Palais [20] and Cerf [5] which asserts that any two orientation preserving imbeddings $i, i' : D^n \rightarrow M$ are related by the equation $i' = f \circ i$, for some diffeomorphism $f : M \rightarrow M$. The proof that $M \# S^n$ is diffeomorphic to M will be left to the reader.

LEMMA 2.2. Let M_1, M'_1 and M_2 be closed and simply connected.² If M_1 is h-cobordant to M'_1 then $M_1 \# M_2$ is h-cobordant to $M'_1 \# M_2$.

Proof. We may assume that the dimension n is ≥ 3 . Let $M_1 + (-M'_1) = bW_1$, where M_1 and $-M'_1$ are deformation retracts of W_1 . Choose a differentiable arc A from a point $p \in M_1$ to a point $p' \in -M'_1$

² This hypothesis is imposed in order to simplify the proof. It could easily be eliminated.

within W_1 so that a tubular neighborhood of this arc is diffeomorphic to $R^n \times [0,1]$. Thus we obtain an imbedding

$$i : R^n \times [0,1] \rightarrow W_1$$

with $i(R^n \times 0) \subset M_1$, $i(R^n \times 1) \subset M'_1$, and $i(0 \times [0,1]) = A$. Now form a manifold W from the disjoint sum

$$(W_1 - A) + (M_2 - i_2(0)) \times [0,1]$$

by identifying $i(tu, s)$ with $i_2((1-t)u) \times s$ for each $0 < t < 1$, $0 \leq s \leq 1$, $u \in S^{n-1}$. Clearly W is a compact manifold bounded by the disjoint sum

$$M_1 \# M_2 + (-(M'_1 \# M_2)) .$$

We must show that both boundaries are deformation retracts of W .

First it is necessary to show that the inclusion map

$$M_1 - p \xrightarrow{j} W_1 - A$$

is a homotopy equivalence. Since $n \geq 3$ it is clear that both of these manifolds are simply connected. Mapping the homology exact sequence of the pair $(M_1, M_1 - p)$ into that of the pair $(W_1, W_1 - A)$ we see that j induces isomorphisms of homology groups, and hence is a homotopy equivalence. Now it follows easily, using a Mayer-Vietoris sequence, that the inclusion

$$M_1 \# M_2 \rightarrow W$$

1. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (Probability of getting 2 heads)
 2. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (Probability of getting 2 tails)

$$P(\text{2 heads or 2 tails}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

3. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (Probability of getting 1 head and 1 tail)
 4. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (Probability of getting 1 tail and 1 head)

$$P(\text{1 head and 1 tail}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

5. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (Probability of getting 3 heads)
 6. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 3 tails)
 7. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 2 heads and 1 tail)

$$P(\text{2 heads and 1 tail}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

8. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 2 tails and 1 head)
 9. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 1 head and 2 tails)

$$P(\text{1 head and 2 tails}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

10. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 3 tails)
 11. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 1 head and 2 tails)

$$P(\text{2 heads and 1 tail}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

12. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 2 tails and 1 head)
 13. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (Probability of getting 3 tails)

$$P(\text{3 tails}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

is a homotopy equivalence; hence that $M_1 \# M_2$ is a deformation retract of W . Similarly $M'_1 \# M_2$ is a deformation retract of W , which completes the proof of Lemma 2.2.

LEMMA 2.3. A simply connected manifold M is h-cobordant to the sphere S^n if and only if M bounds a contractible manifold.

(Here the hypothesis of simple connectivity cannot be eliminated.)

Proof. If $M + (-S^n) = bW$ then filling in a disk D^{n+1} we obtain a manifold W' with $bW' = M$. If S^n is a deformation retract of W then it clearly follows that W' is contractible.

Conversely if $M = bW'$ with W' contractible, then removing the interior of an imbedded disk we obtain a simply connected manifold W with $bW = M + (-S^n)$. Mapping the homology exact sequence of the pair (D^{n+1}, S^n) into that of the pair (W', W) we see that the inclusion $S^n \rightarrow W$ induces a homology isomorphism; hence S^n is a deformation retract of W . Now applying the Poincaré duality isomorphism

$$H_k(W, M) \simeq H^{n+1-k}(W, S^n)$$

we see that the inclusion $M \rightarrow W$ also induces isomorphisms of homology groups. Since M is simply connected, this completes the proof.

LEMMA 2.4. If M is a homotopy sphere, then $M \# (-M)$ bounds a contractible manifold.

Proof. Let $H^2 \subset D^2$ denote the half-disk consisting of all $(t \sin \theta, t \cos \theta)$ with $0 \leq t \leq 1$, $0 \leq \theta \leq \pi$, and let $\frac{1}{2} D^n \subset D^n$

denote the disk of radius $\frac{1}{2}$. Given an imbedding $i : D^n \rightarrow M$ form W from the disjoint union

$$(M - i(\frac{1}{2} D^n)) \times [0, \pi] + S^{n-1} \times H^2$$

by identifying $i(tu) \times \theta$ with $u \times ((2t-1)\sin \theta, (2t-1)\cos \theta)$ for each $\frac{1}{2} < t \leq 1$, $0 \leq \theta \leq \pi$. (Intuitively we are removing the interior of $i(\frac{1}{2} D^n)$ from M and then "rotating" the result through 180° around the resulting boundary.) It is easily verified that W is a differentiable manifold with $\partial W = M \# (-M)$. Furthermore W contains $M - \text{Interior } i(\frac{1}{2} D^n)$ as deformation retract, and therefore is contractible. This proves Lemma 2.4.

Proof of Theorem 1.1. Let θ_n denote the collection of all n -cobordism classes of homotopy n -spheres. By Lemmas 2.1 and 2.2 there is a well defined, associative, commutative addition operation in θ_n . The sphere S^n serves as zero element. By Lemmas 2.3, 2.4 each element of θ_n has an inverse. Therefore θ_n is an additive group.

Clearly θ_1 is zero. For $n \leq 3$ Munkres [19] and Whitehead [31] have proved that a topological n -manifold has a differentiable structure which is unique up to diffeomorphism. It follows that $\theta_2 = 0$. If the Poincaré hypothesis were proved, it would follow that θ_3 is zero; but at present the structure of θ_3 remains unknown. For $n > 3$ the structure of θ_n will be studied in the following sections.

Addendum. There is a slight modification of the connected sum construction which is frequently useful. Let W_1 and W_2 be

$(n+1)$ -manifolds with boundary. Then the sum $bW_1 \# bW_2$ is the boundary of a manifold W constructed as follows. Let H^{n+1} denote the half-disk consisting of all $x = (x_0, x_1, \dots, x_n)$ with $|x| \leq 1$, $x_0 \geq 0$ and let D^n denote the subset $x_0 = 0$. Choose imbeddings

$$i_q : (H^{n+1}, D^n) \rightarrow (W_q, bW_q), \quad q = 1, 2,$$

so that $i_2 \circ i_1^{-1}$ reverses orientation. Now form W from

$$(W_1 - i_1(0)) + (W_2 - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1-t)u)$ for each $0 < t < 1$, $u \in S^n \cap H^{n+1}$.

It is clear that W is a differentiable manifold with $bW = bW_1 \# bW_2$. Note that W has the homotopy type of $W_1 \vee W_2$: the union with a single point in common.

W will be called the boundary-connected sum of W_1 and W_2 , and denoted by $W = W_1 \pm W_2$.

3. Homotopy spheres are s-parallelizable

Let M be a manifold with tangent bundle $\tau = \tau(M)$, and let ε^1 denote a trivial line bundle over M .

Definition. M will be called s-parallelizable if the Whitney sum $\tau \oplus \varepsilon^1$ is a trivial bundle.³ The bundle $\tau \oplus \varepsilon^1$ will be called the stable tangent bundle of M . It is a stable bundle in the sense of [10]. (The expression s-parallelizable stands for stably parallelizable.)

³ The authors have previously used the term " π -manifold" for an s-parallelizable manifold.

THEOREM 3.1. Every homotopy sphere is s-parallelizable.

In the proof we will use recent results of J. F. Adams [1].

Proof. Let Σ be a homotopy n -sphere. Then the only obstruction to the triviality of $\tau \oplus \varepsilon^1$ is a well defined cohomology class

$$\sigma_n(\Sigma) \in H^n(\Sigma; \pi_{n-1}(SO_{n+1})) = \pi_{n-1}(SO_{n+1}).$$

The coefficient group may be identified with the stable group $\pi_{n-1}(SO)$. But these stable groups have been computed by Bott [4], as follows, for $n \geq 2$:

residue class of $n \bmod 8$:	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0	0	0

(Here \mathbb{Z} , \mathbb{Z}_2 , 0 denote the cyclic groups of order ∞ , 2, 1 respectively.)

Case 1. $n \equiv 3, 5, 6$, or 7 (modulo 8). Then $\pi_{n-1}(SO) = 0$, so that $\sigma_n(\Sigma)$ is trivially zero.

Case 2. $n \equiv 0$ or 4 (modulo 8). Say that $n = 4k$. According to [16], [9], some non-zero multiple of the obstruction class $\sigma_n(\Sigma)$ can be identified with the Pontryagin class $p_k(\tau \oplus \varepsilon^1) = p_k(\tau)$. But the Hirzebruch signature⁴ theorem implies that $p_k[\Sigma]$ is a multiple of the signature $\sigma(\Sigma)$ which is zero

⁴ We will substitute the word "signature" for "index" as used in [7; 13; 16; 17; 26] since this is more in accord with the usage in other parts of mathematics. The signature of the form $x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_{k+l}^2$ is defined to be $\sigma = k - l$.

... ..
... ..
... ..
... ..
... ..

... ..

... ..
... ..
... ..

... ..
... ..

... ..
... ..

... ..
... ..
... ..
... ..
... ..
... ..
... ..
... ..

... ..
... ..
... ..
... ..

since $H^{2k}(\Sigma) = 0$. Therefore every homotopy $4k$ -sphere is s -parallelizable.

Case 3. $n \equiv 1$ or 2 (modulo 8), so that $\pi_{n-1}(SO)$ is cyclic of order 2. For each homotopy sphere Σ the residue modulo 2

$$\sigma_n[\Sigma] \in \pi_{n-1}(SO) \simeq \mathbb{Z}_2$$

is well defined. It follows from an argument of Rohlin that

$$J_{n-1}(\sigma_n) = 0 ,$$

where J_{n-1} denotes the Hopf-Whitehead homomorphism

$$J_{n-1} : \pi_{n-1}(SO_k) \rightarrow \pi_{n+k-1}(S^k)$$

in the stable range $k > n$. But J_{n-1} is a monomorphism for $n \equiv 1$ or 2 (modulo 8). For the case $n = 2$ this fact is well known, and for $n = 9, 10$ it has been proved by Kervaire [11]. For $n = 17, 18$ it has been verified by Kervaire and by Toda in unpublished computations. A proof that J_{n-1} is injective for all $n \equiv 1$ or 2 (modulo 8) has recently been given by J. F. Adams [1]. Now the relation $J_{n-1}(\sigma_n) = 0$ together with the information that J_{n-1} is a monomorphism implies that $\sigma_n = 0$. This finishes the proof of Theorem 3.1.

In conclusion here are two lemmas which clarify the concept of s -parallelizability. The first is essentially due to J. H. C. Whitehead [32].

LEMMA 3.3. Let M be an n -dimensional submanifold of S^{n+k} , $n < k$. Then M is s -parallelizable if and only if its normal bundle is trivial.

LEMMA 3.4. A connected manifold with non-vacuous boundary is s -parallelizable if and only if it is parallelizable.

The proofs will be based on the following lemma. (Compare Milnor [17, Lemma 4].)

Let ξ be a k -dimensional vector space bundle over an n -dimensional complex, $k > n$.

LEMMA 3.5. If the Whitney sum of ξ with a trivial bundle ε^r is trivial then ξ itself is trivial.

Proof. We may assume that $r = 1$, and that ξ is oriented. An isomorphism $\xi \oplus \varepsilon^1 \approx \varepsilon^{k+1}$ gives rise to a bundle map f from ξ to the bundle γ^k of oriented k -planes in $(k+1)$ -space. Since the base space of ξ has dimension n , and since the base space of γ^k is the sphere S^k , $k > n$, it follows that f is null-homotopic; and hence that ξ is trivial.

Proof of Lemma 3.3. Let τ , ν denote the tangent and normal bundles of M . Then $\tau \oplus \nu$ is trivial hence $(\tau \oplus \varepsilon^1) \oplus \nu$ is trivial. Applying Lemma 3.5 the conclusion follows.

Proof of Lemma 3.4. This follows by a similar argument. The hypothesis on the manifold guarantees that every map into a sphere of the same dimension is null-homotopic.

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry, no matter how small, should be recorded to ensure the integrity of the financial data. This includes not only sales and purchases but also expenses and income. The document also mentions the need for regular audits to verify the accuracy of the records.

In the second part, the author describes the various methods used to collect and analyze data. This includes the use of surveys, interviews, and focus groups. The document also discusses the importance of using statistical tools to interpret the data and identify trends. The author notes that while data collection is a critical step, the analysis and interpretation of the data are equally important.

The third part of the document focuses on the challenges of data collection and analysis. It discusses the potential for bias in data collection and the importance of using a representative sample. The document also mentions the challenges of dealing with missing data and the need for careful attention to detail in the analysis process.

Finally, the document concludes with a summary of the key findings and recommendations. It emphasizes the need for ongoing monitoring and evaluation to ensure that the data collection and analysis process remains effective and efficient. The author also suggests that future research should focus on developing new methods for data collection and analysis to improve the accuracy and reliability of the results.

4. Which homotopy spheres bound parallelizable manifolds?

Define a subgroup $bP_{n+1} \subset \theta_n$ as follows. A homotopy n -sphere M represents an element of bP_{n+1} if and only if M is the boundary of a parallelizable manifold. We will see that this condition depends only on the h -cobordism class of M , and that bP_{n+1} does form a subgroup. The object of this section will be to prove the following

THEOREM 4.1. The quotient group θ_n/bP_{n+1} is finite.

Proof. Given an s -parallelizable closed manifold M of dimension n , choose an imbedding

$$i : M \longrightarrow S^{n+k}$$

with $k > n+1$. Such an imbedding exists and is unique up to differentiable isotopy. By Lemma 3.3 the normal bundle of M is trivial. Now choose a specific field ϕ of normal k -frames. Then the Pontryagin-Thom construction yields a map

$$p(M, \phi) : S^{n+k} \longrightarrow S^k .$$

(See Pontryagin [21, pp. 41-57], Thom [28].) The homotopy class of $p(M, \phi)$ is a well defined element of the stable homotopy group

$$\pi_n = \pi_{n+k}(S^k) .$$

Allowing the normal frame field ϕ to vary we obtain a set of elements

$$p(M) = \{p(M, \phi)\} \subset \pi_n .$$

THEORY OF THE EARTH AND ITS HISTORY

The theory of the earth and its history is a branch of geology which deals with the origin and development of the earth and its various parts. It is a science which seeks to explain the processes which have shaped the earth and its features, and to determine the time and sequence of these processes. The theory of the earth and its history is based on the study of the earth's rocks and fossils, and on the principles of geology. It is a science which is constantly developing, as new discoveries are made and new theories are proposed.

The theory of the earth and its history is a branch of geology which deals with the origin and development of the earth and its various parts. It is a science which seeks to explain the processes which have shaped the earth and its features, and to determine the time and sequence of these processes. The theory of the earth and its history is based on the study of the earth's rocks and fossils, and on the principles of geology. It is a science which is constantly developing, as new discoveries are made and new theories are proposed.

THE EARTH'S HISTORY

The earth's history is the story of the earth's development from its origin to the present. It is a story which is constantly unfolding, as new discoveries are made and new theories are proposed. The earth's history is a story of change and development, and it is a story which is constantly being rewritten. The earth's history is a story which is constantly unfolding, as new discoveries are made and new theories are proposed.

THE EARTH'S DEVELOPMENT

The earth's development is the process by which the earth has changed from its origin to the present. It is a process which is constantly unfolding, as new discoveries are made and new theories are proposed. The earth's development is a process of change and development, and it is a process which is constantly being rewritten.

THE EARTH'S FEATURES

THE EARTH'S HISTORY

THE EARTH'S DEVELOPMENT

LEMMA 4.2. The subset $p(M) \subset \pi_n$ contains the zero element of π_n if and only if M bounds a parallelizable manifold.

Proof. If $M = bW$ with W parallelizable then the imbedding $i : M \rightarrow S^{n+k}$ can be extended to an imbedding $W \rightarrow D^{n+k+1}$, and W has a field ψ of normal k -frames. We set $\phi = \psi|_M$. Now the Pontryagin-Thom map $p(M, \phi) : S^{n+k} \rightarrow S^k$ extends over D^{n+k+1} , hence is null-homotopic.

Conversely if $p(M, \phi) \simeq 0$ then M bounds a manifold $W \subset D^{n+k+1}$, where ϕ extends to a field ψ of normal frames over W . It follows from Lemmas 3.3 and 3.4 that W is parallelizable. This completes the proof of Lemma 4.2.

LEMMA 4.3. If M_0 is h-cobordant to M_1 then $p(M_0) = p(M_1)$.

Proof. If $M_0 + (-M_1) = bW$ we choose an imbedding of W in $S^{n+k} \times [0,1]$ so that $M_q \rightarrow S^{n+k} \times (q)$ for $q = 0,1$. Then a normal frame field ϕ_q on M_q extends to a normal frame field ψ on W which restricts to some normal frame field ϕ_{1-q} on M_{1-q} . Clearly (W, ψ) gives rise to a homotopy between $p(M_0, \phi_0)$ and $p(M_1, \phi_1)$.

LEMMA 4.4. If M and M' are s-parallelizable then

$$p(M) + p(M') \subset p(M \# M') \subset \pi_n.$$

Proof. Start with the disjoint sum

$$M \times [0,1] + M' \times [0,1]$$

and join the boundary components $M \times 1$ and $M' \times 1$ together, as described in the addendum at the end of §2. Thus we obtain a manifold W bounded by the disjoint sum

$$(M \# M') + (-M) + (-M') .$$

Note that W has the homotopy type of $M \vee M'$, the union with a single point in common.

Choose an imbedding of W in $S^{n+k} \times [0,1]$ so that $(-M)$ and $(-M')$ go into well separated submanifolds of $S^{n+k} \times 0$, and so that $M \# M'$ goes into $S^{n+k} \times 1$. Given fields ϕ and ϕ' of normal k -frames on $(-M)$ and $(-M')$, it is not hard to see that there exists an extension defined throughout W . Let ψ denote the restriction of this field to $M \# M'$. Then clearly $p(M, \phi) + p(M', \phi')$ is homotopic to $p(M \# M', \psi)$. This completes the proof.

LEMMA 4.5. The set $p(S^n) \subset \pi_n$ is a subgroup of the stable homotopy group π_n . For any homotopy sphere Σ the set $p(\Sigma)$ is a coset of this subgroup $p(S^n)$. Thus the correspondence $\Sigma \rightarrow p(\Sigma)$ defines a homomorphism p' from θ_n to the quotient group $\pi_n/p(S^n)$.

Proof. Combining Lemma 4.4 with the identities

$$(1) \quad S^n \# S^n = S^n, \quad (2) \quad S^n \# \Sigma = \Sigma, \quad (3) \quad \Sigma \# (-\Sigma) \sim S^n$$

we obtain

$$(1) \quad p(S^n) + p(S^n) \subset p(S^n)$$

which shows that $p(S^n)$ is a subgroup of π_n ;

$$(2) \quad p(S^n) + p(\Sigma) \subset p(\Sigma)$$

which shows that $p(\Sigma)$ is a union of cosets of this subgroup; and

$$(3) \quad p(\Sigma) + p(-\Sigma) \subset p(S^n)$$

which shows that $p(\sum)$ must be a single coset. This completes the proof of Lemma 4.5.

By Lemma 4.2 the kernel of $p' : \theta_n \rightarrow \pi_n/p(S^n)$ consists exactly of all h -cobordism classes of homotopy n -spheres which bound parallelizable manifolds. Thus these elements form a group which we will denote by $bP_{n+1} \subset \theta_n$. It follows that θ_n/bP_{n+1} is isomorphic to a subgroup of $\pi_n/p(S^n)$. Since π_n is finite (Serre [24]), this completes the proof of Theorem 4.1.

Remarks. The subgroup $p(S^n) \subset \pi_n$ can be described in more familiar terms as the image of the Hopf-Whitehead homomorphism

$$J_n : \pi_n(SO_k) \rightarrow \pi_{n+k}(S^k) .$$

(See Kervaire [9, p. 349].) Hence $\pi_n/p(S^n)$ is the cokernel of J_n . The actual structure of these groups for $n \leq 8$ is given in the following table. For details, and for higher values of n , the reader is referred to Part II of this paper.

n	1	2	3	4	5	6	7	8
π_n	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{240}	$\mathbb{Z}_2 + \mathbb{Z}_2$
$\pi_n/p(S^n)$	0	\mathbb{Z}_2	0	0	0	\mathbb{Z}_2	0	\mathbb{Z}_2
θ_n/bP_{n+1}	0	0	0	0	0	0	0	\mathbb{Z}_2

The prime $q \geq 3$ first divides the order of θ_n/bP_{n+1} for $n = 2q(q-1) - 2$.

Using Theorem 4.1 the proof of the main theorem (Theorem 1.2) stating that θ_n is finite for $n \neq 3$ reduces now to proving that bP_{n+1} is finite for $n \neq 3$.

We will prove that the group bP_{n+1} is zero for n even (§§5, 6) and is finite cyclic for n odd, $n \neq 3$, (see §§7, 8). The first few groups can be given as follows

n	1	3	5	7	9	11	13	15	17	19
order of bP_{n+1}	1	?	1	28	2	992	1	8128	2	130,816

(Again see Part II for details.) The cyclic group bP_{n+1} has order 1 or 2 for $n \equiv 1 \pmod{4}$ but the order grows more than exponentially for $n \equiv 3 \pmod{4}$.

5. Spherical modifications

This section, and §6 which follows, will prove that the groups bP_{2k+1} are zero.⁵ That is:

THEOREM 5.1. If a homotopy sphere of dimension $2k$ bounds an s -parallelizable manifold M , then it bounds a contractible manifold M_1 .

For the case $k = 1$ this assertion is clear since every homotopy 2-sphere is actually diffeomorphic to S^2 . The proof for

⁵ An independent proof of this theorem has been given by C. T. C. Wall [29].

1. 在 1949 年 10 月 1 日，即中华人民共和国成立之日，毛泽东在天安门城楼上向全国人民发表了著名的“开国大典”讲话。在这篇讲话中，毛泽东首先向全国人民宣告：“中华人民共和国中央人民政府今天成立了！”接着，他向全国人民提出了“中国人民从此站起来了”的口号，并号召全国人民“团结起来，为新中国而奋斗”。这篇讲话是新中国历史上具有划时代意义的文献，它不仅宣告了新中国的诞生，也标志着中国历史翻开了新的一页。

2. 在 1954 年 9 月，第一届全国人民代表大会第一次会议在北京召开。在这次会议上，毛泽东向全国人民提出了“四个现代化”的目标，即“把我国建设成为一个伟大的社会主义国家”。这一目标的提出，标志着中国开始进入全面建设社会主义的新时期。

3. 在 1956 年 9 月，中国共产党第八次全国代表大会在北京召开。在这次会议上，毛泽东向全国人民提出了“百花齐放，百家争鸣”的方针，即“在艺术问题上实行百花齐放，在学术问题上实行百家争鸣”。这一方针的提出，标志着中国开始进入社会主义建设的新时期。

4. 在 1958 年 5 月，中国共产党八届十二中全会在北京召开。在这次会议上，毛泽东向全国人民提出了“大跃进”运动，即“以钢为纲，全面跃进”。这一运动的提出，标志着中国开始进入“大跃进”时期。

5. 在 1960 年 6 月，中国共产党八届十一中全会在北京召开。在这次会议上，毛泽东向全国人民提出了“三面红旗”政策，即“总路线、大跃进、人民公社”。这一政策的提出，标志着中国开始进入“三面红旗”时期。

6. 在 1966 年 5 月，中国共产党八届十二中全会在北京召开。在这次会议上，毛泽东向全国人民提出了“文化大革命”运动，即“无产阶级专政下继续革命”。这一运动的提出，标志着中国开始进入“文化大革命”时期。

7. 在 1976 年 9 月，中国共产党九届十二中全会在北京召开。在这次会议上，毛泽东向全国人民提出了“两个凡是”方针，即“凡是毛主席说的，我们就坚决拥护；凡是中央定的，我们就坚决执行”。这一方针的提出，标志着中国开始进入“两个凡是”时期。

$k > 1$ will be based on the technique of "spherical modifications". (See Wallace [30], Milnor [15; 17]⁶.)

Definition. Let M be a differentiable manifold of dimension $n = p + q + 1$ and let

$$\phi : S^p \times D^{q+1} \rightarrow M$$

be a differentiable imbedding. Then a new differentiable manifold $M' = \chi(M, \phi)$ is formed from the disjoint sum

$$(M - \phi(S^p \times 0)) + D^{p+1} \times S^q$$

by identifying $\phi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$, $0 < t \leq 1$. We will say that M' is obtained from M by the spherical modification $\chi(\phi)$. Note that the boundary of M' is equal to the boundary of M .

In order to prove Theorem 5.1 we will show that the homotopy groups of M can be completely killed by a sequence of such spherical modifications. The effect of a single modification $\chi(\phi)$ on the homotopy groups of M can be described as follows.

Let $\lambda \in \pi_p M$ denote the homotopy class of the map $\phi|_{S^p \times 0}$ from $S^p \times 0$ to M .

LEMMA 5.2. The homotopy groups of M' are given by

$$\pi_i M' \simeq \pi_i M \quad \text{for } i < \min(p, q),$$

and

$$\pi_p M' \simeq \pi_p M / \lambda$$

providing that $p < q$; where \wedge denotes a certain subgroup of π_p^M containing λ .

The proof is straightforward. (Compare [17, Lemma 2].)

Thus if $p < q$ (that is, if $p \leq n/2 - 1$), the effect of the modification $\chi(\phi)$ is to kill the homotopy class λ .

Now suppose that some homotopy class $\lambda \in \pi_p^M$ is given.

LEMMA 5.3. If M^n is s-parallelizable and if $p < n/2$, then the class λ is represented by some imbedding $\phi : S^p \times D^{n-p} \rightarrow M$.

Proof. (Compare [17, Lemma 3].) Since $n \geq 2p+1$ it follows from a well known theorem of Whitney that λ can be represented by an imbedding

$$\phi_0 : S^p \rightarrow M.$$

It follows from Lemma 3.5 that the normal bundle of $\phi_0 S^p$ in M is trivial. Hence ϕ_0 can be extended to the required imbedding $S^p \times D^{n-p} \rightarrow M$.

Thus Lemmas 5.2 and 5.3 assert that spherical modifications can be used to kill any required element $\lambda \in \pi_p^M$ providing that $p \leq n/2 - 1$. There is one danger however. If the imbedding ϕ is chosen badly then the modified manifold $M' = \chi(M, \phi)$ may no longer be s-parallelizable. However the following was proven in [17]. Again let $n \geq 2p+1$.

LEMMA 5.4. The imbedding $\phi : S^p \times D^{n-p} \rightarrow M$ can be chosen within its homotopy class so that the modified manifold $\chi(M, \phi)$ will also be s-parallelizable.

For the proof, the reader may either refer to [17, Theorem 2], or make use of the sharper Lemma 6.2 which will be proved below.

Now combining Lemmas 5.2, 5.3, 5.4 one obtains the following.
(Compare [17, p. 46].)

THEOREM 5.5. Let M be a compact, connected s -parallelizable manifold of dimension $n \geq 2k$. By a sequence of spherical modifications on M one can obtain an s -parallelizable manifold M_1 which is $(k-1)$ -connected.

Recall that $bM_1 = bM$.

Proof. Choosing a suitable imbedding $\phi: S^1 \times D^{n-1} \rightarrow M$ one can obtain an s -parallelizable manifold $M' = \chi(M, \phi)$ such that $\pi_1 M'$ is generated by fewer elements than $\pi_1 M$. Thus after a finite number of steps one can obtain a manifold M'' which is 1-connected. Now, after a finite number of steps, one can obtain an s -parallelizable manifold M''' which is 2-connected, and so on until we obtain a $(k-1)$ -connected manifold. This proves Theorem 5.5.

In order to prove 5.1, where $\dim M = 2k+1$, we must carry this argument one step further obtaining a manifold M_1 which is k -connected. It will then follow from the Poincaré duality theorem that M_1 is contractible.

The difficulty in carrying out this program is that Lemma 5.2 is no longer available. Thus if $M' = \chi(M, \phi)$ where ϕ imbeds $S^k \times D^{k+1}$ in M , the group $\pi_k M'$ may actually be larger than $\pi_k M$. It is first necessary to describe in detail what happens to $\pi_k M$ under such a modification. Since we may assume that M is $(k-1)$ -connected with $k > 1$, the homotopy group $\pi_k M$ may be replaced by homology group $H_k M = H_k(M; \mathbb{Z})$.

LEMMA 5.6. Let $M' = \chi(M, \phi)$ where ϕ imbeds $S^k \times D^{k+1}$ in M , and let

$$M_0 = M - (\text{interior } \phi(S^k \times D^{k+1})) .$$

Then there is a commutative diagram

$$\begin{array}{ccccccc}
 & & & H_{k+1}^{M'} & & & \\
 & & & \downarrow \cdot \lambda' & & & \\
 & & & Z & & & \\
 & & & \downarrow \varepsilon & \searrow \lambda & & \\
 H_{k+1}^M & \xrightarrow{\cdot \lambda} & Z & \xrightarrow{\varepsilon'} & H_k^{M_0} & \xrightarrow{i} & H_k^M \longrightarrow 0 \\
 & & \searrow \lambda' & & \downarrow i' & & \\
 & & & & H_k^{M'} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

such that the horizontal and vertical sequences are exact. It follows that the quotient group $H_k^M / \lambda(Z)$ is isomorphic to $H_k^{M'} / \lambda'(Z)$.

Here the following notations are to be understood. The symbol λ denotes the element of H_k^M which corresponds to the homotopy class $\phi|S^k \times 0$, and λ also denotes the homomorphism

the \mathbb{R}^n space, the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space

the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space

the \mathbb{R}^n space

the \mathbb{R}^n space

the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space is a linear space.

the \mathbb{R}^n space is a linear space.

$Z \rightarrow H_k M$ which carries 1 into λ . On the other hand $\cdot \lambda : H_{k+1} M \rightarrow Z$ denotes the homomorphism which carries each $\mu \in H_{k+1} M$ into the intersection number $\mu \cdot \lambda$. The symbols λ' and $\cdot \lambda'$ are to be interpreted similarly. The element $\lambda' \in H_k M'$ corresponds to the homotopy class $\phi' | 0 \times S^k$ where

$$\phi' : D^{k+1} \times S^k \rightarrow M'$$

denotes the canonical imbedding.

Proof of Lemma 5.6. As horizontal sequence take the exact sequence

$$H_{k+1} M \rightarrow H_{k+1}(M, M_0) \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \rightarrow H_k(M, M_0)$$

of the pair (M, M_0) . By excision the group $H_j(M, M_0)$ is isomorphic to

$$H_j(S^k \times D^{k+1}, S^k \times S^k) \simeq \begin{cases} Z & \text{for } j = k+1 \\ 0 & \text{for } j < k+1. \end{cases}$$

Thus we obtain

$$H_{k+1} M \rightarrow Z \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \rightarrow 0$$

as asserted. Since a generator of $H_{k+1}(M, M_0)$ clearly has intersection number ± 1 with the cycle $\phi(S^k \times 0)$ which represents λ , it follows that the homomorphism $H_{k+1} M \rightarrow Z$ can be described as the homomorphism $\mu \rightarrow \mu \cdot \lambda$. The element $\varepsilon' = \varepsilon'(1) \in H_k M_0$ can clearly be described as the homology class corresponding to the "meridian" $\phi(x_0 \times S^k)$ of the torus $\phi(S^k \times S^k)$, where x_0 denotes a base point in S^k .

1. The first part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time. The paper then discusses the various factors that have influenced the development of the English language, including the influence of other languages, the influence of social and cultural changes, and the influence of technological advances.

2. The second part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

3. The third part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

4. The fourth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

5. The fifth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

6. The sixth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

7. The seventh part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

8. The eighth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

9. The ninth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

10. The tenth part of the paper discusses the importance of the study of the history of the English language. It is noted that the English language has a long and rich history, and it is important to understand its development over time.

The vertical exact sequence is obtained in a similar way.

Thus $\varepsilon = \varepsilon(1) \in H_k M_0$ is the homology class of the "parallel" $\phi(S^k \times x_0)$ of the torus. Clearly $i(\varepsilon) \in H_k M$ is equal to the homology class λ of $\phi(S^k \times 0)$. Similarly $i'(\varepsilon') = \lambda'$.

From this diagram the isomorphisms

$$H_k M / \lambda(Z) \simeq H_k M_0 / \varepsilon(Z) + \varepsilon'(Z) \simeq H_k M' / \lambda'(Z)$$

are apparent. This completes the proof of 5.6.

As an application suppose that one chooses an element $\lambda \in H_k M$ which is primitive in the sense that $\mu \cdot \lambda = 1$ for some $\mu \in H_{k+1} M$. It follows that

$$i : H_k M_0 \longrightarrow H_k M$$

is an isomorphism, and hence that

$$H_k M' \simeq H_k M / \lambda(Z) .$$

Thus:

ASSERTION. Any primitive element of $H_k M$ can be killed by a spherical modification.

In order to apply this assertion we assume the following:

Hypothesis. M is a compact, s -parallelizable manifold of dimension $2k+1$, $k > 1$, and is $(k-1)$ -connected. The boundary bM is either vacuous or a homology sphere.

This hypothesis will be assumed for the rest of § 5 and for §6.

LEMMA 5.7. Subject to this hypothesis the homology group $H_k M$ can be reduced to its torsion subgroup by a sequence of spherical modifications. The modified manifold M_1 will still satisfy the hypothesis.

Proof. Suppose that $H_k M \simeq Z \oplus \dots \oplus Z \oplus T$ where T is the torsion subgroup. Let λ generate one of the infinite cyclic summands. Using the Poincaré duality theorem one sees that $\mu_1 \cdot \lambda = 1$ for some element $\mu_1 \in H_{k+1}(M, bM)$. But the exact sequence

$$H_{k+1} M \rightarrow H_{k+1}(M, bM) \rightarrow H_k(bM) = 0$$

shows that μ_1 can be lifted back to $H_{k+1} M$. Therefore λ is primitive, and can be killed by a modification. After finitely many such modifications, one obtains a manifold M_1 with $H_k M_1 \simeq T \subset H_k M$. This completes the proof of 5.7.

Let us specialize to the case k even. Let M be as above, and let $\phi : S^k \times D^{k+1} \rightarrow M$ be any imbedding.

LEMMA 5.8. If k is even then the modification $\chi(\phi)$ necessarily changes the k -th Betti number of M .

The proof will be based on the following lemma. (See Kervaire [8, Formula (8.8)].)

Let F be a fixed field and let W be an orientable homology manifold of dimension $2r$. Define the semi-characteristic $e^*(bW; F)$ to be the following residue class modulo 2:

$$e^*(bW; F) \equiv \sum_{i=0}^{r-1} \text{rank } H_i(bW; F) \pmod{2}.$$

LEMMA 5.9. The rank of the bilinear pairing

$$H_r(W;F) \otimes H_r(W;F) \rightarrow F$$

given by the intersection number is congruent modulo 2 to $e^*(bW;F)$ plus the Euler characteristic $e(W)$.

[For the convenience of the reader, here is a proof.

Consider the exact sequence

$$H_r W \xrightarrow{h} H_r(W, bW) \rightarrow H_{r-1}(bW) \rightarrow \dots \rightarrow H_0(W, bW) \rightarrow 0 ,$$

where the coefficient group F is to be understood. A counting argument shows that the rank of the indicated homomorphism h is equal to the alternating sum of the ranks of the vector spaces to the right of h in this sequence. Reducing modulo 2 and using the identity

$$\text{rank } H_i(W, bW) = \text{rank } H_{2r-i} W$$

this gives

$$\begin{aligned} \text{rank } h &\equiv \sum_{i=0}^{r-1} \text{rank } H_i(bW) + \sum_{i=0}^{2r} \text{rank } H_i W \\ &\equiv e^*(bW;F) + e(W) \pmod{2} . \end{aligned}$$

But the rank of

$$h : H_r W \rightarrow H_r(W, bW) \simeq \text{Hom}_F(H_r W, F)$$

is just the rank of the intersection pairing. This completes the proof.]

Proof of 5.8. First suppose that M has no boundary. As shown in [15] or [17] the manifolds M and $M' = \chi(M, \phi)$, suitably

oriented, together bound a manifold $W = W(M, \phi)$ of dimension $2k+2$. For the moment, since no differentiable structure on W is needed, we can simply define W to be the union

$$(M \times [0, 1]) \cup (D^{k+1} \times D^{k+1})$$

where it is understood that $S^k \times D^{k+1}$ is to be pasted onto $M \times 1$ by the imbedding ϕ . Clearly W is a topological manifold with:

$$\partial W = M \times 0 + M' \times 1 .$$

Note that W has the homotopy type of M with a $(k+1)$ -cell attached. Since the dimension $2k+1$ of M is odd, this means that the Euler characteristic

$$e(W) = e(M) + (-1)^{k+1} = (-1)^{k+1} .$$

Since k is even the intersection pairing

$$H_{k+1}(W; \mathbb{Q}) \otimes H_{k+1}(W; \mathbb{Q}) \rightarrow \mathbb{Q}$$

is skew symmetric, hence has even rank. Therefore Lemma 5.9 (with rational coefficients) asserts that

$$e^*(M + M'; \mathbb{Q}) + (-1)^{k+1} \equiv 0 \pmod{2} ,$$

and hence that

$$e^*(M; \mathbb{Q}) \not\equiv e^*(M'; \mathbb{Q}) .$$

But $H_i M \simeq H_i M' \simeq 0$ for $0 < i < k$, so this implies that

$$\text{rank } H_k(M; \mathbb{Q}) \neq \text{rank } H_k(M'; \mathbb{Q}) .$$

This proves 5.8 providing that M has no boundary.

If M is bounded by a homology sphere, then attaching a cone over ∂M one obtains a homology manifold M_* without boundary. The above argument now shows that

$$\text{rank } H_k(M_*; \mathbb{Q}) \neq \text{rank } H_k(M'_*; \mathbb{Q}) .$$

Therefore the modification $\chi(\phi)$ changes the rank of $H_k(M; \mathbb{Q})$ in this case also. This completes the proof of 5.8.

It is convenient at this point to insert an analogue of 5.8 which will only be used later. (See end of §6.) Let M be as above, with k even or odd, and let $W = W(M, \phi)$.

LEMMA 5.10. Suppose that every mod 2 homology class

$$\xi \in H_k(W; \mathbb{Z}_2)$$

has self-intersection number $\xi \cdot \xi = 0$. Then the modification $\chi(\phi)$ necessarily changes the rank of the mod 2 homology group $H_k(M; \mathbb{Z}_2)$.

The proof is completely analogous to that of 5.8. The hypothesis, $\xi \cdot \xi = 0$ for all ξ , guarantees that the intersection pairing

$$H_k(W; \mathbb{Z}_2) \otimes H_k(W; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

will have even rank.

We now return to the case k even.

Proof of Theorem 5.1 for k even. According to 5.6 we can assume that $H_k M$ is a torsion group. Choose

$$\phi : S^k \times D^{k+1} \rightarrow M$$

as in 5.4 so as to represent a non-trivial $\lambda \in H_k M$.

According to 5.6 we have

$$H_k M / \lambda(Z) \simeq H_k M' / \lambda'(Z) .$$

Since the group $\lambda(Z)$ is finite it follows from 5.8 that $\lambda'(Z)$ must be infinite. Thus the sequence

$$0 \rightarrow Z \xrightarrow{\lambda'} H_k M' \rightarrow H_k M' / \lambda'(Z) \rightarrow 0$$

is exact. It follows that the torsion subgroup of $H_k M'$ maps monomorphically into $H_k M' / \lambda'(Z)$; and hence is definitely smaller than $H_k M$. Now according to 5.7 we can perform a modification on M' so as to obtain a new manifold M'' with

$$H_k M'' \simeq \text{Torsion subgroup of } H_k M' < H_k M .$$

Thus in two steps one can replace $H_k M$ by a smaller group.

Iterating this construction a finite number of times, the group $H_k M$ can be killed completely. This completes the proof of Theorem 5.1 for k even.

6. Framed spherical modifications

This section will complete the proof of Theorem 5.1 by taking care of the case k odd. This case is somewhat more difficult than the case k even (which was handled in §5) since it is necessary to choose the imbeddings ϕ more carefully, taking particular care not to lose s -parallelizability in the process.

Before starting on the proof it is convenient to sharpen the concepts of s-parallelizable manifold, and of spherical modification.

Definition. A framed manifold (M, f) will mean a differentiable manifold M together with a fixed trivialization f of the stable tangent bundle $\tau_M \oplus \epsilon_M$.

Now consider a spherical modification $\chi(\phi)$ of M . Recall that M and $M' = \chi(M, \phi)$ together bound a manifold

$$W = (M \times [0, 1]) \cup (D^{p+1} \times D^{q+1})$$

where the subset $S^p \times D^{q+1}$ of $D^{p+1} \times D^{q+1}$ is pasted onto $M \times 1$ by the imbedding ϕ . (Compare Milnor [16].) It is easy to give W a differentiable structure, except along the "corner" $S^p \times S^q$. A neighborhood of this corner will be "diffeomorphic" with $S^p \times S^q \times Q$ where

$$Q \subset \mathbb{R}^2$$

denotes the three-quarter disk consisting of all $(r \cos \theta, r \sin \theta)$ with $0 \leq r < 1$, $0 \leq \theta \leq 3\pi/2$. In order to "straighten" this corner, map Q onto the half-disk H , consisting of all $(r \cos \theta', r \sin \theta')$ with $0 \leq r < 1$, $0 \leq \theta' \leq \pi$; by setting $\theta' = 2\theta/3$. Now carrying the differentiable structure of H back to Q , this makes Q into a differentiable manifold. Carrying out the same transformation on the neighborhood of $S^p \times S^q$, this makes $W = W(M, \phi)$ into the required differentiable manifold. Note that both boundaries of W get the correct differential structures.

the first of these is the fact that the
the second is the fact that the
the third is the fact that the
the fourth is the fact that the
the fifth is the fact that the
the sixth is the fact that the
the seventh is the fact that the
the eighth is the fact that the
the ninth is the fact that the
the tenth is the fact that the

the eleventh is the fact that the
the twelfth is the fact that the
the thirteenth is the fact that the
the fourteenth is the fact that the
the fifteenth is the fact that the
the sixteenth is the fact that the
the seventeenth is the fact that the
the eighteenth is the fact that the
the nineteenth is the fact that the
the twentieth is the fact that the

the twenty-first is the fact that the
the twenty-second is the fact that the
the twenty-third is the fact that the
the twenty-fourth is the fact that the
the twenty-fifth is the fact that the
the twenty-sixth is the fact that the
the twenty-seventh is the fact that the
the twenty-eighth is the fact that the
the twenty-ninth is the fact that the
the thirtieth is the fact that the

Now identify M with $M \times 0 \subset W$ and identify the stable tangent bundle $\tau_M \oplus \varepsilon_M$ with the restriction $\tau_W|_M$. Thus a framing f of M determines a trivialization of $\tau_W|_M$.

Definition. A framed spherical modification $\chi(\phi, F)$ of the framed manifold (M, f) will mean a spherical modification $\chi(\phi)$ of M together with a trivialization F of the tangent bundle of W , satisfying the condition

$$F|_M = f .$$

Note that the modified manifold $M' = \chi(M, \phi)$ automatically acquires a framing

$$f' = F|_{M'} .$$

It is only necessary to identify $\tau_W|_{M'}$ with the stable tangent bundle $\tau_M \oplus \varepsilon_M$. To do this we identify the positive direction in ε_M with the outward normal direction in $\tau_W|_{M'}$.

The following question evidently arises. Given a modification $\chi(\phi)$ of M and a framing f of M , does f extend to a trivialization F of τ_W ? The obstructions to such an extension lie in the cohomology groups

$$H^{r+1}(W, M; \pi_r(SO_{n+1})) \simeq \begin{cases} \pi_p(SO_{n+1}) & \text{for } r = p \\ 0 & \text{for } r \neq p . \end{cases}$$

Thus the only obstruction to extending f is a well defined class

$$\gamma(\phi) \in \pi_p(SO_{n+1}) .$$

The modification $\chi(\phi)$ can be framed if and only if this obstruction $\gamma(\phi)$ is zero.

Now consider the following alteration of the imbedding ϕ .

Let

$$\alpha : S^p \rightarrow SO_{q+1}$$

be a differentiable map, and define

$$\phi_\alpha : S^p \times D^{q+1} \rightarrow M$$

by

$$\phi_\alpha(u, v) = \phi(u, v \cdot \alpha(u))$$

where the dot denotes the usual action of SO_{q+1} on D^{q+1} . Clearly ϕ_α is an imbedding which represents the same homotopy class $\lambda \in \pi_p M$ as ϕ .

LEMMA 6.1. The obstruction $\gamma(\phi_\alpha)$ depends only on $\gamma(\phi)$ and on the homotopy class (α) of α . In fact

$$\gamma(\phi_\alpha) = \gamma(\phi) + s_*(\alpha)$$

where $s_* : \pi_p(SO_{q+1}) \rightarrow \pi_p(SO_{n+1})$ is induced by the inclusion $s : SO_{q+1} \rightarrow SO_{n+1}$.

Proof. (Compare [17], proof of Theorem 2.) Let W_α be the manifold constructed as W above, now using ϕ_α . There is a natural differentiable imbedding

$$i_\alpha : D^{p+1} \times_{\text{int.}} D^{q+1} \rightarrow W_\alpha ,$$

and $i_\alpha|_{S^p \times D^{q+1}}$ coincides with $\phi_\alpha : S^p \times D^{q+1} \rightarrow M$ followed by the inclusion $M \rightarrow M \times 1 \subset W_\alpha$.

$\gamma(\phi_\alpha)$ is the obstruction to extending $f|_{\phi_\alpha}(S^p \times 0)$ to a trivialization of $\tau(W_\alpha)$ restricted to $i_\alpha(D^{p+1} \times 0)$. Let $t^{n+1} = e^{p+1} \times e^{q+1}$ be the standard framing on $D^{p+1} \times D^{q+1}$. Then $i'_\alpha(t^{n+1})$ is a trivialization of the tangent bundle of W_α restricted to $i_\alpha(D^{p+1} \times D^{q+1})$, and $\gamma(\phi_\alpha)$ is the homotopy class of the map $g : S^p \rightarrow SO_{n+1}$, where $g(u)$ is the matrix $\langle f^{n+1}, i'_\alpha(t^{n+1}) \rangle$ at $\phi_\alpha(u, 0)$.

Since $i_\alpha|_{D^{p+1} \times 0}$ is independent of α , and $i_\alpha|_{S^p \times D^{q+1}} = \phi_\alpha$, we have

$$i'_\alpha(t^{n+1}) = i'(e^{p+1}) \times \phi'_\alpha(e^{q+1})$$

at every point $(u, 0) \in S^p \times D^{q+1}$.

Since

$$\phi'_\alpha(e^{q+1}) = \phi'(e^{q+1}) \cdot \alpha(u)$$

at $(u, 0)$, it follows that

$$i'_\alpha(t^{n+1}) = i'(t^{n+1}) \cdot s(\alpha) .$$

Hence

$$\langle f^{n+1}, i'_\alpha(t^{n+1}) \rangle = \langle f^{n+1}, i'(t^{n+1}) \rangle \cdot s(\alpha)$$

and the lemma follows.

Now suppose (as usual) that $p \leq q$. Then the homomorphism

$$s_* : \pi_p(SO_{q+1}) \rightarrow \pi_p(SO_{n+1})$$

is onto. Hence α can be chosen so that

$$\gamma(\phi_\alpha) = \gamma(\phi) + s_*(\alpha)$$

1. The first part of the paper is devoted to the study of the

properties of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

2. In the second part, we consider the function $g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

3. The third part of the paper is devoted to the study of the

properties of the function $h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$.

4. In the fourth part, we consider the function $k(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

(1.1.1)

5. The fifth part of the paper is devoted to the study of the

properties of the function

$$l(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{R}$. In the sixth part, we consider the function

$m(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{R}$. In the seventh part, we consider the function

$$o(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{C}$.

$$(1.1.2) \quad p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{R}$. In the eighth part, we consider the function

$$q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$r(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$s(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$t(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is zero. Thus we obtain:

LEMMA 6.2. Given $\phi: S^p \times D^{q+1} \rightarrow M$ with $p \leq q$, a map α can be chosen so that the modification $\chi(\phi_\alpha)$ can be framed.

In particular, it follows that the manifold $\chi(M, \phi_\alpha)$ will be s -parallelizable. Thus we have proved Lemma 5.4 in a sharpened form.

We note however that α is not always uniquely determined. In the case $p = q = k$ odd the homomorphism

$$s_* : \pi_k(SO_{k+1}) \rightarrow \pi_k(SO_{n+1})$$

has an infinite cyclic kernel. This freedom in the choice of α will be the basis of the proof of 5.1 for k odd.

Let us study the homology of the manifold

$$M'_\alpha = \chi(M, \phi_\alpha) ,$$

where ϕ is now chosen, by Lemma 6.2, so that the spherical modification $\chi(\phi)$ can be framed. Clearly the deleted manifold

$$M_0 = M - (\text{interior } \phi_\alpha(S^k \times D^{k+1}))$$

does not depend on the choice of α . Furthermore the meridian $\phi_\alpha(x_0 \times S^k)$ of the torus $\phi_\alpha(S^k \times S^k) \subset M_0$ does not depend on the choice of α ; hence the homology class

$$\varepsilon' \in H_k M_0$$

does not depend on α . On the other hand the parallel $\phi_\alpha(S^k \times x_0)$ does depend on α . In fact it is clear that the homology class

$\varepsilon_\alpha \in H_k^M O$ of this parallel is given by

$$\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$$

where the homomorphism

$$j_* : \pi_k(SO_{k+1}) \longrightarrow \mathbb{Z} \simeq \pi_k(S^k)$$

is induced by the canonical map

$$\rho \xrightarrow{j} x_0 \cdot \rho$$

from SO_{k+1} to S^k .

The spherical modification $\chi(\phi_\alpha)$ can still be framed provided α is an element of the kernel of

$$s_* : \pi_k(SO_{k+1}) \longrightarrow \pi_k(SO_{n+1}) .$$

Identifying the stable group $\pi_k(SO_{n+1})$ with the stable group $\pi_k(SO_{k+2})$, there is an exact sequence

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{s_*} \pi_k(SO_{k+2})$$

associated with the fibration $SO_{k+2}/SO_{k+1} = S^{k+1}$. It is well known that the composition

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{j_*} \pi_k(S^k)$$

carries a generator of $\pi_{k+1}(S^{k+1})$ onto twice a generator $\pi_k(S^k)$, providing that k is odd. Therefore the integer $j_*(\alpha)$ can be any multiple of 2.

Let us study the effect of replacing ε by $\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$ on the homology of the modified manifold. Consider the exact sequence

$$0 \longrightarrow Z \xrightarrow{\varepsilon'} H_k M_0 \xrightarrow{i} H_k M \longrightarrow 0$$

of 5.6, where i carries ε into an element λ of order $\ell > 1$.

Evidently $\ell\varepsilon$ must be a multiple of ε' , say:

$$\ell\varepsilon + \ell'\varepsilon' = 0.$$

Since ε' is not a torsion element, these two elements can satisfy no other relation. Since $\varepsilon_\alpha = \varepsilon + j_*(\alpha)\varepsilon'$ it follows that

$$\ell\varepsilon_\alpha + (\ell' - \ell j(\alpha))\varepsilon' = 0.$$

Now using the sequence

$$0 \longrightarrow Z \xrightarrow{\varepsilon_\alpha} H_k M_0 \xrightarrow{i_\alpha} H_k M'_\alpha \longrightarrow 0$$

we see that the inclusion homomorphism i_α carries ε' into an element

$$\lambda'_\alpha \in H_k M'_\alpha$$

of order $|\ell' - \ell j(\alpha)|$. Since $H_k M'_\alpha / \lambda'_\alpha(Z)$ is isomorphic to $H_k M / \lambda(Z)$ we see that the group $H_k M'_\alpha$ is smaller than $H_k M_\alpha$ if and only if:

$$0 < |\ell' - \ell j(\alpha)| < \ell.$$

But $j(\alpha)$ can be any even integer. Thus $j(\alpha)$ can be chosen so that

$$-\ell < \ell' - \ell j(\alpha) \leq \ell.$$

This choice of $j(\alpha)$ will guarantee an improvement except in the special case where ℓ' happens to be divisible by ℓ .

Our progress so far can be summarized as follows.

LEMMA 6.3. Let M be a framed $(k-1)$ -connected manifold of dimension $2k+1$ with k odd, $k > 1$, such that $H_k M$ is finite. Let $\chi(\phi, F)$ be a framed modification of M which replaces the element $\lambda \in H_k M$ of order $\ell > 1$ by an element $\lambda' \in H_k M'$ of order $\pm \ell'$. If $\ell' \not\equiv 0 \pmod{\ell}$ then it is possible to choose $(\alpha) \in \pi_k(SO_{k+1})$ so that the modification $\chi(\phi_\alpha)$ can still be framed, and so that the group $H_k M'_\alpha$ is definitely smaller than $H_k M$.

Thus one must study the residue class of ℓ' modulo ℓ . Recall the definition of linking numbers. (Compare Seifert-Threlfall [23, §77].) Let $\lambda \in H_p M$, $\mu \in H_q M$ be homology classes of finite order, with $\dim M = p+q+1$. Consider the homology sequence

$$\dots \longrightarrow H_{q+1}(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H_q M \xrightarrow{i_*} H_q(M; \mathbb{Q}) \longrightarrow \dots$$

associated with the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Since λ is of finite order, $i_* \lambda = 0$ and $\lambda = \beta(v)$ for some $v \in H_{q+1}(M; \mathbb{Q}/\mathbb{Z})$. The pairing

$$\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by multiplication induces a pairing

$$H_p M \otimes H_{q+1}(M; \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

3. The first of these is the fact that the system is not in equilibrium.

The second is the fact that the system is not in equilibrium.

The third is the fact that the system is not in equilibrium.

The fourth is the fact that the system is not in equilibrium.

The fifth is the fact that the system is not in equilibrium.

The sixth is the fact that the system is not in equilibrium.

The seventh is the fact that the system is not in equilibrium.

The eighth is the fact that the system is not in equilibrium.

The ninth is the fact that the system is not in equilibrium.

The tenth is the fact that the system is not in equilibrium.

The eleventh is the fact that the system is not in equilibrium.

The twelfth is the fact that the system is not in equilibrium.

The thirteenth is the fact that the system is not in equilibrium.

The fourteenth is the fact that the system is not in equilibrium.

The fifteenth is the fact that the system is not in equilibrium.

The sixteenth is the fact that the system is not in equilibrium.

The seventeenth is the fact that the system is not in equilibrium.

The eighteenth is the fact that the system is not in equilibrium.

The nineteenth is the fact that the system is not in equilibrium.

The twentieth is the fact that the system is not in equilibrium.

The twenty-first is the fact that the system is not in equilibrium.

The twenty-second is the fact that the system is not in equilibrium.

defined by the intersection of homology classes. We denote this pairing by a dot.

Definition. The linking number $L(\lambda, \mu)$ is the rational number modulo 1 defined by

$$L(\lambda, \mu) = \nu \cdot \mu .$$

This linking number is well defined, and satisfies the symmetry relation

$$L(\mu, \lambda) + (-1)^{pq} L(\lambda, \mu) = 0 .$$

(Compare Seifert and Threlfall.)

LEMMA 6.4. The ratio ℓ'/ℓ modulo 1 is, up to sign, equal to the self-linking number $L(\lambda, \lambda)$.

Proof. Since

$$\ell \varepsilon + \ell' \varepsilon' = 0$$

in $H_k M_0$ we see that the cycle $\ell \varepsilon + \ell' \varepsilon'$ on bM_0 bounds a chain c in M_0 . Let $c_1 = \phi(x_0 \times D^{k+1})$ denote the cycle in $\phi(S^k \times D^{k+1}) \subset M$ with boundary ε' . Then the chain $c - \ell' c_1$ has boundary $\ell \varepsilon$; hence $(c - \ell' c_1)/\ell$ has boundary ε , representing the homology class λ in $H_k M$. Taking the intersection of this chain with $\phi(S^k \times 0)$, representing λ , we obtain $\pm \ell'/\ell$, since c is disjoint and c_1 has intersection number ∓ 1 . Thus $L(\lambda, \lambda) = \pm \ell'/\ell \bmod 1$.

Now if $L(\lambda, \lambda) \neq 0$ then $\ell' \not\equiv 0 \pmod{\ell}$ hence the class λ can be replaced by an element of smaller order under a spherical modification. Hence, unless $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k M$, this group can be simplified.

1. The first step in the process of the development of a new product is the identification of a market need. This is done by conducting market research and analyzing the needs and wants of potential customers.

2. The second step is the development of a concept. This involves creating a detailed description of the product and its features, and determining the target market for the product.

3. Development

3. The third step is the development of a prototype. This involves creating a physical model of the product that can be used to test the design and make any necessary adjustments.

4. Testing

4. The fourth step is the testing of the prototype. This involves conducting a series of tests to determine the product's performance and reliability.

5. The fifth step is the production of the final product. This involves manufacturing the product in large quantities and distributing it to the market.

6. Distribution

6. The sixth step is the distribution of the product. This involves getting the product into the hands of the target market.

7. The seventh step is the evaluation of the product. This involves monitoring the product's performance and making any necessary adjustments.

8. The eighth step is the promotion of the product. This involves creating a marketing plan and implementing it to promote the product to the target market.

9. The ninth step is the maintenance of the product. This involves ensuring that the product is kept in good condition and that any necessary repairs are made.

10. The tenth step is the disposal of the product. This involves ensuring that the product is disposed of properly and that any necessary recycling is done.

11. The eleventh step is the evaluation of the product's success. This involves determining whether the product has met its goals and whether it is profitable.

12. The twelfth step is the conclusion of the process. This involves summarizing the results of the process and making any necessary adjustments.

LEMMA 6.5. If $H_k M$ is a torsion group, with $L(\lambda, \lambda) = 0$ for every $\lambda \in H_k M$, and if k is odd, then this group $H_k M$ must be a direct sum of cyclic groups of order 2.

Proof. The relation

$$L(\eta, \xi) + (-1)^{pq} L(\xi, \eta) = 0$$

with $p = q \equiv 1 \pmod{2}$ implies that

$$L(\eta, \xi) = L(\xi, \eta) .$$

Now if self-linking numbers are all zero, the identity

$$L(\xi + \eta, \xi + \eta) = L(\xi, \xi) + L(\eta, \eta) + L(\xi, \eta) + L(\eta, \xi)$$

implies that

$$2L(\xi, \eta) = 0$$

for all ξ and η . But, according to the Poincaré duality theorem for torsion groups (see [23, page 245]), L defines a completely orthogonal pairing

$$T_p M \otimes T_q M \rightarrow \mathbb{Q}/\mathbb{Z} .$$

Hence the identity $L(2\xi, \eta) = 0$ for all η implies that $2\xi = 0$. This proves Lemma 6.5.

It follows that, by a sequence of modifications, one can reduce $H_k M$ to a group of the form $\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2 = s\mathbb{Z}_2$.

Now let us apply Lemma 5.8. Since the modification $\chi(\phi_\alpha)$ is framed, the corresponding manifold $W = W(M, \phi_\alpha)$ is parallelizable. It follows from the formulas of Wu that the Steenrod operation

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..

$$Sq^{k+1} : H^{k+1}(W; bW; Z_2) \rightarrow H^{2k+2}(W, bW; Z_2)$$

is zero. (See Kervaire [8, Lemma (7.9)].) Hence every $\xi \in H_{k+1}(W; Z_2)$ has self-intersection number $\xi \cdot \xi = 0$. Thus, according to 5.10, the modification $\chi(\phi_\alpha)$ changes the rank of $H_k(M; Z_2)$.

But the effect of $\chi(\phi_\alpha)$ on $H_k(M; Z)$, providing that α is chosen properly, will be to replace the element λ of order $\ell = 2$ by an element λ'_α of order ℓ'_α where

$$-2 < \ell'_\alpha \leq 2, \quad \ell'_\alpha \equiv 0 \pmod{2}.$$

Thus ℓ'_α must be 0 or 2. Now using the sequence

$$0 \rightarrow Z_{\ell'_\alpha} \rightarrow H_k M'_\alpha \rightarrow H_k M'_\alpha / \lambda'_\alpha(Z) \rightarrow 0,$$

where the right hand group is isomorphic to $(s-1)Z_2$, we see that $H_k M'_\alpha$ is given by one of the following:

$$H_k M'_\alpha \simeq \begin{cases} Z + (s-1)Z_2, \\ Z_2 + (s-1)Z_2, \\ Z + (s-2)Z_2, \text{ or} \\ Z_4 + (s-2)Z_2. \end{cases}$$

But the first two possibilities cannot occur, since they do not change the rank of $H_k(M; Z_2)$. In the remaining two cases, a further modification will replace $H_k M'_\alpha$ by a group which is definitely smaller than $H_k M$. Thus in all cases $H_k M$ can be replaced by a smaller group by a sequence of framed modifications.

This completes the proof of 5.1. Actually we have proved the following result which is slightly sharper.

THEOREM 6.6. Let M be a compact, framed manifold of dimension $2k+1$, $k > 1$, such that bM is either vacuous or a homology sphere. By a sequence of framed modifications M can be reduced to a k -connected manifold M_1 .

If bM is vacuous then the Poincaré duality theorem implies that M_1 is a homotopy sphere. If bM is a homology sphere then M_1 is contractible.

The proof of 6.6 is contained in the above discussion, providing that M is connected. But using [17, Lemma 2'] it is easily seen that a disconnected manifold can be connected by framed modifications. This completes the proof.

§7. The groups bP_{2k}

The next two sections will prove that the groups bP_{2k} are finite cyclic for $k \neq 2$. In fact for k odd the group bP_{2k} has at most two elements. For $k = 2m \neq 2$ we will see in part II that bP_{4m} is a cyclic group of order

$$2^{2m-2}(2^{2m-1} - 1) \text{ numerator } (4B_m/m),$$

where B_m denotes the m -th Bernoulli number.⁷

The proofs will be based on the following.

⁷ This expression for the order of bP_{4m} relies on provisional results of J. F. Adams [2].

LEMMA 7.1. Let M be a $(k-1)$ -connected manifold of dimension $2k$, $k \geq 3$, and suppose that $H_k M$ is free abelian with basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ where

$$\lambda_i \cdot \lambda_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}$$

for all i, j (where δ_{ij} denotes a Kronecker delta). Suppose further that every imbedded sphere in M which represents a homology class in the subgroup generated by $\lambda_1, \dots, \lambda_r$ has trivial normal bundle. Then $H_k M$ can be killed by a sequence of spherical modifications.

Proof. According to [17, Lemma 6] or Haefliger [6] any homology class in $H_k M$ can be represented by a differentiably imbedded sphere.

Remark: It is at this point that the hypothesis $k \geq 3$ is necessary. Our methods break down completely for the case $k = 2$ since a homology class in $H_2(M^4)$ need not be representable by a differentiably imbedded sphere. (Compare Kervaire and Milnor [13].)

Choose an imbedding $\phi_0 : S^k \rightarrow M$ so as to represent the homology class λ_r . Since the normal bundle is trivial, ϕ_0 can be extended to an imbedding $\phi : S^k \times D^k \rightarrow M$. Let $M' = \chi(M, \phi)$ denote the modified manifold, and let

$$M_0 = M - \text{Interior } \phi(S^k \times D^k) = M' - \text{Interior } \phi'(D^{k+1} \times S^{k-1}).$$

The argument now proceeds just as in [17, p. 54]. There is a diagram

... ..
... ..
... ..

... ..
... ..
... ..
... ..
... ..

... ..
... ..
... ..
... ..

... ..
... ..
... ..
... ..
... ..

... ..
... ..
... ..
... ..
... ..

... ..
... ..
... ..

$$\begin{array}{ccccccc}
 & & Z & & & & \\
 & & \downarrow & \searrow \lambda_r & & & \\
 0 & \longrightarrow & H_k M_0 & \longrightarrow & H_k M & \xrightarrow{\cdot \lambda_r} & Z \longrightarrow H_{k-1} M_0 \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & H_k M' & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

where the notation and the proof is similar to that of Lemma 5.6. Since $\mu_r \cdot \lambda_r = 1$ it follows that $H_{k-1} M_0 = 0$. From this fact one easily proves that M_0 and M' are $(k-1)$ -connected. The group $H_k M_0$ is isomorphic to the subgroup of $H_k M$ generated by $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_{r-1}\}$. The group $H_k M'$ is isomorphic to a quotient group of $H_k M_0$. It has basis $\{\lambda'_1, \dots, \lambda'_{r-1}, \mu'_1, \dots, \mu'_{r-1}\}$ where each λ'_i corresponds to a coset

$$\lambda_i + \lambda_r Z \subset H_k M,$$

and each μ'_j corresponds to a coset $\mu_j + \lambda_r Z$.

The manifold M' also satisfies the hypothesis of 7.1. In order to verify that

$$\lambda'_i \cdot \lambda'_j = 0, \quad \lambda'_i \cdot \mu'_j = \delta_{ij},$$

note that each λ'_i or μ'_j can be represented by a sphere imbedded in M_0 and representing the homology class λ_i or μ_j of M . Thus the intersection numbers in M' are the same as those in M . In order to verify that any imbedded sphere with homology class

$n_1\lambda'_1 + \dots + n_{r-1}\lambda'_{r-1}$ has trivial normal bundle, note that any such sphere can be pushed off of $\phi'(0 \times S^{k-1})$, and hence can be deformed into M_0 . It will then represent a homology class

$$(n_1\lambda_1 + \dots + n_{r-1}\lambda_{r-1}) + n_r\lambda_r \in H_k M,$$

and thus will have trivial normal bundle.

Iterating this construction r times, the result will be a k -connected manifold. This completes the proof of Lemma 7.1.

Now consider an s -parallelizable manifold M of dimension $2k$, bounded by a homology sphere. By Theorem 5.5 we can assume that M is $(k-1)$ -connected. Using the Poincaré duality theorem it follows that $H_k M$ is free abelian, and that the intersection number pairing

$$H_k M \otimes H_k M \rightarrow \mathbb{Z}$$

has determinant ± 1 . The argument now splits up into three cases.

Case 1. Let $k = 3$ or 7 . (Compare [17, Theorem 4'].) Since k is odd the intersection pairing is skew symmetric. Hence there exists a "symplectic" basis for $H_k M$: that is a basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ with

$$\lambda_i \cdot \lambda_j = \mu_i \cdot \mu_j = 0, \quad \lambda_i \cdot \mu_j = -\delta_{ij}.$$

Since $\pi_{k-1}(SO_k) = 0$ for $k = 3, 7$, any imbedded k -sphere will have trivial normal bundle. Thus Lemma 7.1 implies that $H_k M$ can be killed. Since an analogous result for $k = 1$ is easily obtained, this proves:

LEMMA 7.2. The groups bp_2 , bp_6 , and bp_{14} are zero.

Case 2. k is odd, but $k \neq 1, 3, 7$. Again one has a symplectic basis; but the normal bundle of an imbedded sphere is not necessarily trivial. This case will be studied in §8.

Case 3. k is even, say $k = 2m$. Then the following is true. (Compare [17, Theorem 4].)

LEMMA 7.3. Let M be a framed manifold of dimension $4m > 4$, bounded by a homology sphere⁸. The homotopy groups of M can be killed by a sequence of framed spherical modifications if and only if the signature $\sigma(M)$ is zero.

Since a proof of 7.3 is essentially given in [17] we will only give an outline here.

In one direction the lemma follows from the assertion that $\sigma(M)$ is invariant under spherical modifications. (See [17, p. 41].) The fact that M has a boundary does not matter here, since we can adjoin a cone over the boundary, thus obtaining a closed homology manifold with the same signature.)

Conversely suppose that $\sigma(M) = 0$. We may assume that M is $(k-1)$ -connected. Since the quadratic form $\lambda \rightarrow \lambda \cdot \lambda$ has determinant ± 1 and signature zero, it is possible to choose a basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ for $H_k M$ so that $\lambda_i \cdot \lambda_j = 0$, $\lambda_i \cdot \mu_j = \delta_{ij}$. The proof is analogous to that of [17, Lemma 9], but somewhat simpler since we do not put any restriction on $\mu_i \cdot \mu_j$. For any imbedded sphere with homology class $\lambda = n_1 \lambda_1 + \dots + n_r \lambda_r$, the self-intersection number $\lambda \cdot \lambda$ is zero. Therefore, according to [17, Lemma 7], the normal bundle is trivial.

⁸ This lemma is of course also true if ∂M is vacuous. In this case the signature $\sigma(M)$ is necessarily zero, by Hirzebruch's signature theorem.

Thus M satisfies the hypothesis of 7.1. It follows that $H_k M$ can be killed by spherical modifications. Since the homomorphism

$$\pi_k(SO_k) \rightarrow \pi_k(SO_{2k+1})$$

is onto for k even, it follows from Lemma 6.2 that we need use only framed spherical modifications. This completes the proof of 7.3.

LEMMA 7.4. For each $k = 2m$ there exists a parallelizable manifold M_0 whose boundary bM_0 is the ordinary $(4m-1)$ -sphere, such that the signature $\sigma(M_0)$ is non-zero.

Proof. According to Milnor and Kervaire [18, p. 457] there exists a closed "almost parallelizable" $4m$ -manifold whose signature is non-zero. Removing the interior of an imbedded $4m$ -disk from this manifold, we obtain the required parallelizable manifold M_0 .

Now consider the collection of all $4m$ -manifolds M_0 which are s -parallelizable, and are bounded by the $(4m-1)$ -sphere. Clearly the corresponding signatures $\sigma(M_0) \in \mathbb{Z}$ form a group under addition. Let $\sigma_m > 0$ denote the generator of this group.

THEOREM 7.5. Let \sum_1 and \sum_2 be homotopy spheres of dimension $4m-1$, $m > 1$, which bound s -parallelizable manifolds M_1 and M_2 respectively. Then \sum_1 is h -cobordant to \sum_2 if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m}.$$

Proof. First suppose that

$$\sigma(M_1) = \sigma(M_2) + \sigma(M_0).$$

... of the
... ..

... ..

... ..
... ..

... ..
... ..

... ..
... ..
... ..
... ..

... ..
... ..

... ..
... ..
... ..

... ..

... ..

... ..

Form the connected sum along the boundary

$$M = (-M_1) \pm M_2 \pm M_0$$

as in §2; with boundary

$$bM = -\Sigma_1 \# \Sigma_2 \# S^{4m-1} \approx -\Sigma_1 \# \Sigma_2.$$

Since

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) + \sigma(M_0) = 0$$

it follows from 7.3 that $bM = -\Sigma_1 \# \Sigma_2$ belongs to the trivial h-cobordism class. Therefore Σ_1 is h-cobordant to Σ_2 .

Conversely let W be an h-cobordism between $-\Sigma_1 \# \Sigma_2$ and the sphere S^{4m-1} . Pasting W onto $-M_1 \pm M_2$ along the common boundary $-\Sigma_1 \# \Sigma_2$, we obtain a differentiable manifold M bounded by the sphere S^{4m-1} . Since M is clearly s-parallelizable, we have

$$\sigma(M) \equiv 0 \pmod{\sigma_m}.$$

But

$$\sigma(M) = \sigma(-M_1 \pm M_2) = -\sigma(M_1) + \sigma(M_2).$$

Therefore

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m},$$

which completes the proof.

COROLLARY 7.6. The group bP_{4m} , $m > 1$, is isomorphic to a subgroup of the cyclic group of order σ_m . Hence bP_{4m} is finite cyclic.

The proof is evident.

Discussion and computations. In Part II we will see that bP_{4m} is cyclic of order precisely $\sigma_m/8$. In fact a given integer σ

... ..

...

...

...

...

...

...

...

...

...

...

...

...

...

...

...

...

...

...

...

occurs as $\sigma(M)$ for some s -parallelizable M bounded by a homotopy sphere if and only if

$$\sigma \equiv 0 \pmod{8}.$$

The following equality is proved in [18, p. 457]:

$$\sigma_m = 2^{2m-1}(2^{2m-1} - 1)B_m j_m a_m / m,$$

where B_m denotes the m -th Bernoulli number, j_m denotes the order of the cyclic group

$$J(\pi_{4m-1}(SO)) \subset \pi_{4m-1},$$

and a_m equals 1 or 2 according as m is even or odd. Thus bP_{4m} is cyclic of order

$$(1) \quad \sigma_m / 8 = 2^{2m-4}(2^{2m-1} - 1)B_m j_m a_m / m.$$

According to recent work of J. F. Adams and H. Toda, the integer j_m is precisely equal to the denominator of $B_m/4m$. (Compare [18, Theorem 4].) Therefore

$$B_m j_m a_m / 4m = a_m \text{ numerator } (B_m/4m) = \text{numerator } (4B_m/m)$$

where the last equality holds since the denominator of B_m is divisible by 2 but not 4. Thus bP_{4m} is cyclic of order

$$(2) \quad \sigma_m / 8 = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator } (4B_m/m).$$

One can also give a formula for the order of the full group θ_{4m-1} . In Part II we will see that θ_{4m-1}/bP_{4m} is isomorphic to

$\pi_{4m-1}/J(\pi_{4m-1}(SO))$. (Compare §4.) Together with formula (1) above this implies that:

$$\text{order } \theta_{4m-1} = (\text{order } \pi_{4m-1}) 2^{2m-4} (2^{2m-1} - 1) B_m a_m / m .$$

§8. A cohomology operation

Let $2 \leq k \leq n-2$ be integers and let (K,L) be a CW-pair which satisfies the following:

Hypothesis. The cohomology groups $H^i(K,L;G)$ vanish for $k < i < n$ and for all coefficient groups G .

Then a cohomology operation

$$\psi : H^k(K,L;Z) \rightarrow H^n(K,L;\pi_{n-1}(S^k))$$

is defined as follows⁹. Let $e^0 \in S^k$ denote a base point and let

$$s \in H^k(S^k, e^0; Z)$$

denote a generator. Then $\psi(c)$ will denote the first obstruction to the existence of a map

$$f : (K,L) \rightarrow (S^k, e^0)$$

satisfying the condition $f^*(s) = c$.

To be more precise let K^r denote the r -skeleton of K . Then given any class

⁹ A closely related operation ϕ_0 has been studied by Kervaire [12]. The operation ϕ_0 would serve equally well for our purposes.

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

... ..

$$x \in H^k(K, L; \mathbb{Z}) \simeq H^k(K^{n-1} \cup L, L; \mathbb{Z})$$

it follows from standard obstruction theory that there exists a map

$$f_x : (K^{n-1} \cup L, L) \rightarrow (S^k, e^0)$$

with $f_x^* s = x$; and that the restriction

$$f_x|_{(K^{n-2} \cup L, L)}$$

is well defined up to homotopy. The obstruction to extending f_x over $K^n \cup L$ is the required class

$$\psi(x) \in H^n(K, L; \pi_{n-1}(S^k)) .$$

LEMMA 8.1. The function

$$\psi : H^k(K, L; \mathbb{Z}) \rightarrow H^n(K, L; \pi_{n-1}(S^k))$$

is well defined, and is natural in the following sense. If the CW-pair (K', L') also satisfies the hypothesis above, then for any map

$$g : (K', L') \rightarrow (K, L)$$

and any $x \in H^k(K, L; \mathbb{Z})$ the identity

$$g^* \psi(x) = \psi g^*(x)$$

is satisfied.

The proof is straightforward. It follows that ψ does not depend on the particular cell structure of the pair (K, L) .

Now let us specialize to the case $n = 2k$.

LEMMA 8.2. The operator ψ satisfies the identity

$$\psi(x+y) = \psi(x) + \psi(y) + [i,i](x \smile y) ,$$

where the last term stands for the image of the class
 $x \smile y \in H^{2k}(K,L;Z)$ under that coefficient homomorphism

$$Z \rightarrow \pi_{2k-1}(S^k)$$

which carries 1 into the Whitehead product class $[i,i]$.

Proof. Let $U = e^0 \cup e^k \cup \{e_i^{2k}\} \cup \{e_j^{2k+1}\} \cup \dots$ denote a complex formed from the sphere S^k by adjoining cells of dimension $\geq 2k$ so as to kill the homotopy groups in dimensions $\geq 2k-1$. Let

$$u \in H^k(U, e^0; Z)$$

be a standard generator. Evidently the functions

$$\psi : H^k U \rightarrow H^{2k}(U; \pi_{2k-1}(S^k))$$

and

$$\psi : H^k(U \times U) \rightarrow H^{2k}(U \times U; \pi_{2k-1}(S^k))$$

are defined. We will first evaluate $\psi(u \times 1 + 1 \times u)$.

The $(2k+1)$ -skeleton of $U \times U$ consists of the union

$$U^{2k+1} \times e^0 \cup e^0 \times U^{2k+1} \cup e^k \times e^k .$$

Therefore the cohomology class $\psi(u \times 1 + 1 \times u) \in H^{2k}(U \times U; \pi_{2k-1}(S^k))$ can be expressed uniquely in the form

$$a \times 1 + 1 \times b + \gamma(u \times u)$$

with $a, b \in H^{2k}(U; \pi_{2k-1}(S^k))$ and $\gamma \in \pi_{2k-1}(S^k)$. Applying 8.1 to the inclusion map

$$U \times e^0 \rightarrow U \times U$$

we see that a must be equal to $\psi(u)$. Similarly b is equal to $\psi(u)$. Applying 8.1 to the inclusion

$$S^k \times S^k \rightarrow U \times U$$

we see that $\psi(s \times 1 + 1 \times s) = \gamma(s \times s)$. But $\psi(s \times 1 + 1 \times s)$ is just the obstruction to the existence of a mapping

$$f : S^k \times S^k \rightarrow S^k$$

satisfying $f(e^0, x) = f(x, e^0) = x$. Therefore γ must be equal to the Whitehead product class $[i, i] \in \pi_{2k-1}(S^k)$. Thus we obtain the identity

$$\begin{aligned} \psi(u \times 1 + 1 \times u) &= \psi(u) \times 1 + 1 \times \psi(u) + [i, i](u \times u) \\ &= \psi(u \times 1) + \psi(1 \times u) + [i, i]((u \times 1) \cup (1 \times u)) . \end{aligned}$$

Now consider an arbitrary CW-pair (K, L) , and two classes $x, y \in H^k(K, L)$. Choose a map

$$g : (K, L) \rightarrow (U \times U, e^0 \times e^0)$$

so that $g^*(u \times 1) = x$, $g^*(1 \times u) = y$. (Such a map can be constructed inductively over the skeletons of K since the obstruction groups $H^1(K, L; \pi_{i-1}(U \times U))$ are all zero.) Then by 8.1:

$$\begin{aligned}
\psi(x+y) &= g^* \psi(u \times 1 + 1 \times u) \\
&= g^* \psi(u \times 1) + g^* \psi(1 \times u) + [i, i] g^* ((u \times 1) + (1 \times u)) \\
&= \psi(x) + \psi(y) + [i, i](x \cup y) .
\end{aligned}$$

This completes the proof of 8.2.

Now let M be a $2k$ -manifold which is $(k-1)$ -connected. Then

$$\psi : H^k(M, bM) \rightarrow H^{2k}(M, bM; \pi_{2k-1}(S^k)) \simeq \pi_{2k-1}(S^k)$$

is defined.

LEMMA 8.3. Let k be odd¹⁰ and let M be s -parallelizable.
Then an imbedded k -sphere in M has trivial normal bundle if and
only if its dual cohomology class $v \in H^k(M, bM)$ satisfies the
condition $\psi(v) = 0$.

Proof. Let N be a closed tubular neighborhood of the imbedded sphere, and let

$$M_0 = M - \text{Interior } N .$$

Then there is a commutative diagram

$$\begin{array}{ccc}
w \in H^k(N, bN) & \xrightarrow{\psi} & H^{2k}(N, bN; \pi_{2k-1}(S^k)) \\
\uparrow \simeq & & \uparrow \simeq \\
H^k(M, M_0) & \xrightarrow{\psi} & H^{2k}(M, M_0; \pi_{2k-1}(S^k)) \\
\downarrow \simeq & & \downarrow \simeq \\
v \in H^k(M, bM) & \xrightarrow{\psi} & H^{2k}(M, bM; \pi_{2k-1}(S^k)) ;
\end{array}$$

¹⁰ This lemma is actually true for k even also.

where a generator w of the infinite cyclic group $H^k(N, bN)$ corresponds to the cohomology class v under the left hand vertical arrows. Thus¹¹

$$\psi(v)[M] = \psi(w)[N] \in \pi_{2k-1}(S^k) .$$

It is clear that the homotopy class $\psi(w)[N]$ depends only on the normal bundle of the imbedded sphere.

The normal bundle is determined by an element v of the group $\pi_{k-1}(SO_k)$. Since M is s -parallelizable, v must belong to the kernel of the homomorphism

$$\pi_{k-1}(SO_k) \rightarrow \pi_{k-1}(SO) .$$

But this kernel is zero for $k = 1, 3, 7$, and is cyclic of order 2 for other odd values of k . The unique non-trivial element corresponds to the tangent bundle of S^k , or equivalently to the normal bundle of the diagonal in $S^k \times S^k$.

Thus if $v \neq 0$ then N can be identified with a neighborhood of the diagonal in $S^k \times S^k$. Then

$$\psi(w)[N] = \psi(s \times 1 + 1 \times s)[S^k \times S^k] = [i, i] \neq 0$$

(assuming that $k \neq 1, 3, 7$). On the other hand if $v = 0$ then $\psi(w)$ is clearly zero. This completes the proof of 8.3.

Henceforth we will assume that k is odd and $\neq 1, 3, 7$. The subgroup of $\pi_{2k-1}(S^k)$ generated by $[i, i]$ will be identified with the standard cyclic group Z_2 . Thus a function

¹¹ The symbol $[M]$ denotes the homomorphism $H^n(M, bM; G) \rightarrow G$ determined by the orientation homology class in $H_n(M, bM; Z)$.

$$\psi_0 : H_k M \rightarrow Z_2$$

is defined by the formula

$$\psi_0(\lambda) = \psi(x)[M]$$

where $x \in H^k(M, bM)$ denotes the Poincaré dual of the homology class λ . Evidently:

$$1) \quad \psi_0(\lambda + \mu) \equiv \psi_0(\lambda) + \psi_0(\mu) + \lambda \cdot \mu \pmod{2}, \text{ and}$$

2) $\psi_0(\lambda) = 0$ if and only if an imbedded sphere representing the homology class λ has trivial normal bundle.

Now assume that bM has no homology in dimensions $k, k-1$, so that the intersection pairing has determinant ± 1 . Then one can choose a symplectic basis for $H_k M$: that is a basis $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ such that

$$\lambda_i \cdot \lambda_j = 0, \quad \mu_i \cdot \mu_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}.$$

Definition. The Arf invariant $c(M)$ is defined to be the residue class¹²

$$\psi_0(\lambda_1)\psi_0(\mu_1) + \dots + \psi_0(\lambda_r)\psi_0(\mu_r) \in Z_2.$$

(Compare [3].) This residue class modulo 2 does not depend on the choice of symplectic basis.

LEMMA 8.4. If $c(M) = 0$ then $H_k M$ can be killed by a sequence of framed spherical modifications.

¹² This coincides with the invariant $\overline{\Phi}(M)$ as defined by Kervaire [12].

44-38861-101

1. *Chlorophyll a* and *Chlorophyll b* contents were determined by the method of Lichtenthaler and Whistler (1973).

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840. 84

• • • •

The proof will depend on Lemma 7.1. Let $\{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r\}$ be a symplectic basis for $H_k M$. By permuting the λ_i and μ_i we may assume that

$$\psi_0(\lambda_i) = \psi_0(\mu_i) = 1 \quad \text{for } i \leq s ,$$

$$\psi_0(\lambda_i) = 0 \quad \text{for } i > s ,$$

where s is an integer between 0 and r . The hypothesis

$$c(M) = \sum \psi_0(\lambda_i) \psi_0(\mu_i) = 0$$

implies that $s \equiv 0 \pmod{2}$.

Construct a new basis $\{\lambda'_1, \dots, \mu'_r\}$ for $H_k M$ by the substitutions

$$\lambda'_{2i-1} = \lambda_{2i-1} + \lambda_{2i} , \quad \lambda'_{2i} = \mu_{2i-1} - \mu_{2i} ,$$

$$\mu'_{2i-1} = \mu_{2i-1} , \quad \mu'_{2i} = \lambda_{2i} ,$$

for $2i \leq s$, with

$$\lambda'_i = \lambda_i , \quad \mu'_i = \mu_i$$

for $i > s$. This new basis is again symplectic, and satisfies the condition:

$$\psi_0(\lambda'_1) = \dots = \psi_0(\lambda'_r) = 0 .$$

For any sphere imbedded in M with homology class

$\lambda = n_1 \lambda'_1 + \dots + n_r \lambda'_r$ the invariant $\psi_0(\lambda)$ is zero, and hence the normal bundle is trivial. Thus the basis $\{\lambda'_1, \dots, \mu'_r\}$ satisfies of Lemma 7.1. Thus $H_k M$ can be killed by spherical modifications.

If M is a framed manifold then it is only necessary to use framed modifications for this construction. This follows from Lemma 6.2, since the homomorphism $\pi_k(SO_k) \rightarrow \pi_k(SO_{2k+1})$ is onto for $k \neq 1, 3, 7$. This completes the proof of 8.4.

THEOREM 8.5. For k odd the group bp_{2k} is either zero or cyclic of order 2.

According to Lemma 7.2 the groups bp_2 , bp_6 and bp_{14} are zero. Thus we may assume that $k \neq 1, 3, 7$.

Let M_1 and M_2 be s -parallelizable and $(k-1)$ -connected manifolds of dimension $2k$, bounded by homotopy spheres. If

$$c(M_1) = c(M_2)$$

we will prove that bM_1 is h -cobordant to bM_2 . This will clearly prove 8.5.

Form the connected-sum-along-the-boundary $M_1 \sharp M_2$. Clearly

$$c(M_1 \sharp M_2) = c(M_1) + c(M_2) = 0.$$

Therefore, according to 8.4, it follows that the boundary

$$b(M_1 \sharp M_2) = bM_1 \# bM_2$$

bounds a contractible manifold. Hence, according to Theorem 1.1 the manifold bM_1 is h -cobordant to $-bM_2$. Since a similar argument shows that bM_2 is h -cobordant to $-bM_2$, this completes the proof.

Remark. It seems plausible that $bp_{2k} \simeq \mathbb{Z}_2$ for all odd k other than $1, 3, 7$; but this is known to be true only for $k = 5$ (compare Kervaire [12]) and $k = 9$.

REFERENCES

- [1] Adams, J. F., Vector fields on spheres, Annals of Math. (to appear).
- [2] Adams, J. F., (in preparation).
- [3] Arf, C., Untersuchungen über quadratische Formen in Körpern der Charakteristik 2, Crelles Math. Journal 183 (1941), 148-167.
- [4] Bott, R., The stable homotopy of the classical groups, Annals of Math. 70 (1954), 313-337.
- [5] Cerf, J., Topologie de certains espaces de plongements, Bull. Soc. Math. France (to appear).
- [6] Haefliger, A., Plongements différentiables de variétés dans variétés, Comm. Math. Helv. 36 (1961), 47-82.
- [7] Hirzebruch, F., Neue topologische Methoden in der algebraischen Geometrie, Springer Verlag 1956.
- [8] Kervaire, M., Relative characteristic classes, Amer. Journal of Math. 79 (1957), 517-558.
- [9] Kervaire, M., An interpretation of G. Whitehead's generalization of the Hopf invariant, Annals of Math. 69 (1959), 345-364.
- [10] Kervaire, M., A note on obstructions and characteristic classes, Amer. Journal of Math. 81 (1959), 773-784.
- [11] Kervaire, M., Some non-stable homotopy groups of Lie groups, Illinois Journal of Math. 4 (1960), 161-169.
- [12] Kervaire, M., A manifold which does not admit any differentiable structure, Comm. Math. Helv. 34 (1960), 257-270.
- [13] Kervaire, M. and Milnor, J., On 2-spheres in 4-manifolds, Proc. Nat. Acad. Sci. 47 (1961), 1651-1657.
- [14] Milnor, J., On manifolds homeomorphic to the 7-sphere, Annals of Math. 64 (1956), 399-405.

- [15] Milnor, J., Differentiable manifolds which are homotopy spheres, Mimeographed Notes, Princeton 1958.
- [16] Milnor, J., Sommes de variétés différentiables et structures différentiables des sphères, Bull. Soc. Math. France 87 (1959), 439-444.
- [17] Milnor, J., A procedure for killing the homotopy groups of differentiable manifolds, Symposia in Pure Math. A.M.S., vol. III (1961), 39-55.
- [18] Milnor, J. and Kervaire, M., Bernoulli numbers, homotopy groups and a theorem of Rohlin, Proc. Int. Congress of Math., Edinburgh 1958.
- [19] Munkres, J., Obstructions to the smoothing of piecewise-homeomorphisms, Bull. Amer. Math. Soc., 65 (1959), 332-334.
- [20] Palais, R., Extending diffeomorphisms, Proc. Amer. Math. Soc. 11 (1960), 274-277.
- [21] Pontryagin, L., Smooth manifolds and their applications in homotopy theory, Trudy Mat. Inst. im. Steklov 45 (1955), and A.M.S. translations, Series 2, 11, 1-114.
- [22] Seifert, H., Konstruktion dreidimensionaler geschlossener Räume, Ber. Verh. Sächs. Akad. Wiss. Leipzig 83 (1931), 26-66.
- [23] Seifert, H. and Threlfall, W., Lehrbuch der Topologie, Teubner Verlag, 1934.
- [24] Serre, J-P., Homologie singulière des espaces fibrés. Applications, Annals of Math. 54 (1951), 425-505.
- [25] Smale, S., Generalized Poincaré conjecture in dimensions greater than four, Annals of Math. 74 (1961), 391-406.
- [26] Smale, S., On the structure of manifolds, Annals of Math. (to appear).

1. The first part of the report is a general introduction to the subject of the study. It discusses the importance of the study and the objectives of the research.

2. The second part of the report is a detailed description of the methodology used in the study. It includes information about the sample size, the data collection methods, and the statistical analysis techniques.

3. The third part of the report is a discussion of the results of the study. It presents the findings of the research and compares them with the previous studies in the field.

4. The fourth part of the report is a conclusion and a list of references. The conclusion summarizes the main findings of the study and provides recommendations for future research. The references list the sources of information used in the study.

5. The fifth part of the report is an appendix containing additional information related to the study. This may include raw data, detailed calculations, or other supporting materials.

6. The sixth part of the report is a bibliography of the literature cited in the study. This provides a comprehensive list of the sources used in the research.

7. The seventh part of the report is a list of figures and tables. This provides a visual representation of the data and results of the study.

8. The eighth part of the report is a list of abbreviations and symbols used in the study. This helps to clarify the meaning of the symbols and abbreviations used throughout the report.

9. The ninth part of the report is a list of acknowledgments. This section is used to thank the individuals and organizations that provided support and assistance during the study.

10. The tenth part of the report is a list of references. This provides a comprehensive list of the sources used in the research.

11. The eleventh part of the report is a list of figures and tables. This provides a visual representation of the data and results of the study.

- [27] Stallings, J., Polyhedral homotopy-spheres, Bull. Amer. Math. Soc. 66 (1960), 485-488.
- [28] Thom, R., Quelques propriétés globales des variétés différentiables, Comm. Math. Helv. 28 (1954), 17-86.
- [29] Wall, C. T. C., Killing the middle homotopy groups of odd dimensional manifolds, (to appear).
- [30] Wallace, A. H., Modifications and cobounding manifolds, Canadian J. Math. 12 (1960), 503-528.
- [31] Whitehead, J. H. C., Manifolds with transverse fields in euclidean space, Annals of Math. 73 (1961), 154-212.
- [32] Whitehead, J. H. C., On the homotopy type of manifolds, Annals of Math. 41 (1940), 825-832.
- [33] Zeeman, C., The generalized Poincaré conjecture, Bull. Amer. Math. Soc. 67 (1961), 270.

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt.$$

It is shown that the function $f(x)$ is continuous and differentiable on the interval $(-\infty, \infty)$.

2. In the second part of the paper, we consider the function $F(x)$ defined by the equation

$$F(x) = \int_0^x \frac{1}{1+t^2} dt + \int_0^x \frac{1}{1+t^4} dt.$$

It is shown that the function $F(x)$ is continuous and differentiable on the interval $(-\infty, \infty)$.

3. In the third part of the paper, we consider the function $G(x)$ defined by the equation

$$G(x) = \int_0^x \frac{1}{1+t^2} dt + \int_0^x \frac{1}{1+t^4} dt + \int_0^x \frac{1}{1+t^6} dt.$$

It is shown that the function $G(x)$ is continuous and differentiable on the interval $(-\infty, \infty)$.

4. In the fourth part of the paper, we consider the function $H(x)$ defined by the equation

OFFICIAL DISTRIBUTION LIST

| | | | |
|--|-------------|--|--------|
| Chief of Naval Research
Navy Department
Washington 25, D.C.
Attn: Code 432
Code 438 | 5
3 | Chief, Bureau of Ships
Navy Department
Washington 25, D.C.
Attn: Library
Code 280 | 2
1 |
| Director
Naval Research Laboratory
Washington 25, D.C.
Attn: Library
Code 6230
Tech. Information
Officer | 2
2
6 | Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
Attn: Tech. Library | |
| Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York | | Director
David Taylor Model Basin
Washington 25, D.C.
Attn: Library
Dr. H. Polachek | 1
1 |
| Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California
Attn: Tech. Library | | U.S. Naval Electronics Laboratory
San Diego 52, California
Attn: Library
Dr. F. A. Sabransky | 1
1 |
| Commanding Officer
Office of Naval Research
Branch Office
495 Summer Street
Boston 10, Massachusetts | | U.S. Naval Weapons Plant
Washington 25, D.C.
Attn: Library | |
| Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco, California | | U.S. Navy Underwater Sound Lab.
Fort Trumbull
New London, Connecticut | |
| Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco, California | | Naval Ordnance Laboratory
White Oak
Silver Spring, Maryland
Attn: Mechanics Division
Library | 1
1 |
| Commanding Officer
Office of Naval Research
Branch Office
Navy No. 100, Fleet Post Office
New York, New York | 40 | U.S. Naval Hydrographic Office
Suitland, Maryland | 2 |
| | | Beach Erosion Board
U.S. Corps of Engineers
Little Falls Road, N.W.
Washington 16, D.C. | |
| | | Superintendent
U.S. Naval Postgraduate School
Monterey, California | |

Figure 1. The effect of the concentration of the *Agrobacterium* suspension on the transformation efficiency of *Agrobacterium* strains.

[illegible]

the 1990s, the number of people in the world who are undernourished has declined from 1.1 billion to 800 million. The number of people who are malnourished has declined from 1.5 billion to 1 billion. The number of people who are obese has increased from 100 million to 300 million. The number of people who are overweight has increased from 100 million to 300 million. The number of people who are obese and overweight has increased from 100 million to 300 million. The number of people who are obese and overweight has increased from 100 million to 300 million.

[illegible][illegible][illegible]

Journal of Management Studies, 19(1), 67-80.

[illegible]

Journal of Management Education 30(6)p. 789-806
© The Author(s) 2006. Reprints and permissions:
<http://www.sagepub.com/journalsPermissions.nav>

the 1990s, the number of people in the world who are under 15 years of age is expected to increase from 1.2 billion to 1.5 billion. The number of people aged 65 and over is expected to increase from 200 million to 350 million. The number of people aged 15 and over is expected to increase from 3.5 billion to 4.5 billion. The number of people aged 15 and over is expected to increase from 3.5 billion to 4.5 billion. The number of people aged 15 and over is expected to increase from 3.5 billion to 4.5 billion.

100-30-11100
100-30-11100
100-30-11100
100-30-11100
100-30-11100

1. The first step is to identify the problem or question that needs to be answered. This involves understanding the context and the specific requirements of the task.

the 1990s, the number of people in the world who are undernourished has declined from 760 million to 600 million. The number of people who are malnourished has declined from 1.1 billion to 800 million. The number of people who are obese has increased from 100 million to 300 million. The number of people who are overweight has increased from 100 million to 300 million. The number of people who are obese and overweight has increased from 100 million to 300 million. The number of people who are obese and overweight has increased from 100 million to 300 million.

[illegible]

Commander
U.S. Naval Weapons Laboratory
Dahlgren, Virginia
Attn: Library

Commanding General
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Library

Commanding General
Wright-Patterson Air Force Base,
Ohio
Attn: Central Air Documents (D13)

Aeronautical Research 1
Lab. 1

Chief
Armed Forces Special Weapons
Project
Washington 25, D.C.

Armed Services Technical
Information Agency
Arlington Hall Station
Arlington 12, Virginia 10

U. S. Department of Commerce
Washington 25, D.C.
Attn: National Hydraulics Lab.

The Computing Laboratory
National Applied Mathematics Lab.
National Bureau of Standards
Washington 25, D.C.

California Institute of Technology
Hydrodynamics Laboratory
Pasadena, California

University of California
Department of Engineering
Berkeley 4, California
Attn: Dr. J.W. Johnson 1
Dr. S. Schaaf 1

Carnegie Institute of Technology
Department of Mathematics
Pittsburgh, Pennsylvania

Chesapeake Bay Institute
The John Hopkins University
121 Maryland Hall
Baltimore 18, Maryland
Attn: Director,
Dr. D. W. Pritchard

University of Chicago
Department of Meteorology
Chicago, Illinois
Attn: Dr. C.G. Rossby

Columbia University
New York 27, New York
Attn: Prof. M.G. Salvadori

Harvard University
Department of Mathematics
Cambridge, Massachusetts
Attn: Prof. G. Birkhoff

Indiana University
Department of Mathematics
Bloomington, Indiana
Attn: Prof. T.Y. Thomas

State University of Iowa
Iowa Institute of Hydraulic
Research
Iowa City, Iowa
Attn: Prof. L. Landweber

Massachusetts Institute of
Technology
Cambridge 38, Massachusetts
Attn: Dr. E. Reissner 1
Dr. C. C. Lin 1

New York University
Department of Meteorology
New York, New York
Attn: Dr. W. J. Pierson, Jr.

Princeton University
Department of Mathematics
Princeton University
Attn: Prof. S. Lefschetz

Rand Corporation
1700 Main Street
Santa Monica, California

Scripps Institute of Oceanography
La Jolla, California
Attn: Dr. W. Munk 1
Dr. R.S. Arthur 1

University of Washington
Oceanographic Department
Seattle 5, Washington
Attn: Dr. T.G. Thompson 1
Dr. Maurice Rattray, Jr. 1

Woods Hole Oceanographic Institute
Woods Hole, Massachusetts
Attn: Dr. C. Iselin

Marine Biological Laboratory
Woods Hole, Massachusetts
Attn: Library

Dr. Milton Rose
Radiation Laboratory
Livermore, California

Dr. Arthur Grad
National Science Foundation
Washington 25, D.C.

Serials Work Room
Main Library, 206
New York University
New York, New York

Dr. C. R. DePrima
California Institute of Technology
1201 East California Street
Pasadena, California

Office of Technical Services
Department of Commerce
Washington 25, D. C.

Department of Mathematics
Stanford University
Stanford, California

Department of Mathematics
Harvard University
Cambridge, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Professor B. Friedman
Department of Mathematics
University of California
Berkeley, California

Director (Code 120)
USN Underwater Sound Reference
Lab.

P.O. Box 8337
Orlando, Florida

Professor A. Erdelyi
Department of Mathematics
California Institute of Technology
Pasadena, California

Professor J. Todd
Department of Mathematics
California Institute of Technology
Pasadena, California

Professor C. H. Wilcox
Department of Mathematics
California Institute of Technology
Pasadena, California

Professor B. Zumino
Department of Physics
New York University
New York, New York

Department of Mathematics
University of California
Berkeley, California

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

... ..
... ..
... ..

Professor H. C. Kranzer
Department of Mathematics
Adelphi College
Garden City, New York

Dr. F. J. Weyl
Research Director, Code 402
Office of Naval Research
Washington 25, D. C.

DATE DUE

| | | | |
|------------|--|--|--|
| JUN 06 '62 | | | |
| AUG 18 '63 | | | |
| MAY 29 '64 | | | |
| NOW 23 '65 | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| GAYLORD | | | |

W 1 11 3 14 17

**N. Y. U. Institute of
Mathematical Sciences**
4 Washington Place
New York 3, N. Y.

